EE3301 Electromagnetic Fields

UNIT I ELECTROSTATICS – I

Sources and effects of electromagnetic fields – Coordinate Systems – Vector fields –Gradient, Divergence, Curl – theorems and applications – Coulomb's Law – Electric field intensity – Field due to discrete and continuous charges – Gauss's law and applications.

UNIT II ELECTROSTATICS – II

Electric potential – Electric field and equipotential plots, Uniform and Non-Uniform field, Utilization factor – Electric field in free space, conductors, dielectrics – Dielectric polarization –Dielectric strength – Electric field in multiple dielectrics – Boundary conditions, Poisson's and Laplace's equations, Capacitance, Energy density, Applications.

UNIT III MAGNETOSTATICS

Lorentz force, magnetic field intensity (H) – Biot–Savart's Law – Ampere's Circuit Law – H due to straight conductors, circular loop, infinite sheet of current, Magnetic flux density (B) – B in free space, conductor, magnetic materials – Magnetization, Magnetic field in multiple media –Boundary conditions, scalar and vector potential, Poisson's Equation, Magnetic force, Torque, Inductance, Energy density, Applications.

UNIT IV ELECTRODYNAMIC FIELDS

Magnetic Circuits – Faraday's law – Transformer and motional EMF – Displacement current -Maxwell's equations (differential and integral form) – Relation between field theory and circuit theory –Applications.

UNIT V ELECTROMAGNETIC WAVES

Electromagnetic wave generation and equations – Wave parameters; velocity, intrinsic impedance, propagation constant – Waves in free space, lossy and lossless dielectrics, conductors- skin depth – Poynting vector – Plane wave reflection and refraction.

TEXT BOOKS:

1. Mathew N. O. Sadiku, 'Principles of Electromagnetics', 6th Edition, Oxford University Press Inc. Asian edition, 2015.
2. William H. Hayt and John A. Buck, 'Engineering Electromagnetics', McGraw Hill Special Indian edition, 2014.
3. Kraus and Fleish, 'Electromagnetics with Applications', McGraw Hill International Editions, Fifth Edition, 2010.

REFERENCES

1. V.V.Sarwate, 'Electromagnetic fields and waves', Second Edition, Newage Publishers, 2018. EE3301 Electromagnetic Fields
2. J.P.Tewari, 'Engineering Electromagnetics – Theory, Problems and Applications', Second Edition, Khanna Publishers 2013.
3. Joseph. A.Edminister, 'Schaum's Outline of Electromagnetics, Fifth Edition (Schaum's Outline Series), McGraw Hill, 2018.

4. S.P.Ghosh, Lipika Datta, 'Electromagnetic Field Theory', First Edition, McGraw Hill Education(India) Private Limited, 2017.
5. K A Gangadhar, 'Electromagnetic Field Theory', Khanna Publishers; Sixteenth Edition Eighth Reprint :2015

Relationship between cartesian and spherical systems

$$\left.\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta \end{aligned}\right\} \text{ spherical to Cartesian}$$

$$\left.\begin{aligned} r &= \sqrt{x^2+y^2+z^2} \\ \theta &= \cos^{-1}\left[\frac{z}{\sqrt{x^2+y^2+z^2}}\right] \\ \phi &= \tan^{-1}(y/x) \end{aligned}\right\} \text{ Cartesian to spherical}$$

Problems: ①

Given the two points, A ($x=2$, $y=3$, $z=-1$) and B ($r=4$, $\theta=25$, $\phi=120^\circ$).

Find the spherical co ordinates of A and cartesian co ordinates of B.

Convert to spherical co ordinates:-

A ($x=2$, $y=3$, $z=-1$)

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{2^2+3^2+(-1)^2}$$

$$r = \sqrt{14} = 3.7416$$

$$\theta = \cos^{-1}\left[\frac{z}{\sqrt{x^2+y^2+z^2}}\right] = \cos^{-1}\left[\frac{-1}{14}\right]$$

$$\theta = 105.5'$$

$$\phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\left(\frac{3}{2}\right) = 56.31$$

Ans: A $(3.7416,\ 105.15^\circ,\ 56.31^\circ)$ in spherical system.

Convert Cartesians to spherical

$B\,(r = 4,\ \theta = 25^\circ,\ \phi = 120^\circ)$

$x = r \sin\theta \cos\phi = 4 \times \sin 25^\circ \times \cos 120^\circ = -0.845$

$y = r \sin\theta \sin\phi = 4 \times \sin 25^\circ \times \sin 120 = 1.464$

$z = r \cos\theta = 4 \times \cos 25 = 3.625$

Ans $B\,(-0.845,\ 1.464,\ 3.625)$ in cartesian System

Transformation of Vector :-

(i) Cartesian to cylinderical

$$\begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

(ii) Cylinderical to cartesian

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix}$$

(iii) Cartesian to spherical

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

(iv) Sperical to cartesian

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

(v) spherical to cylinderical

$$\begin{bmatrix} A_\rho \\ A_\varphi \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\varphi \end{bmatrix}$$

(vi) cylinderical to spherical.

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\varphi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\varphi \\ A_z \end{bmatrix}$$

Let $\vec{A} = A_x a_x + A_y a_y + A_z a_z$

Problem ②

Transform $4\bar{a}_x - 2\bar{a}_y - 4\bar{a}_z$ at $(2,3,5)$ to cylinderical co ordinates.

$\vec{A} = 4\bar{a}_x - 2\bar{a}_y - 4\bar{a}_z$ at $(2,3,5)$

$A_r = \bar{A}\cdot\bar{a}_r = 4(\bar{a}_x\cdot\bar{a}_r) - 2(\bar{a}_y\cdot\bar{a}_r) - 4(\bar{a}_z\cdot\bar{a}_r)$

$\boxed{A_r = 4\cos\varphi - 2\sin\varphi}$ — ①

$A_\varphi = \bar{A}\cdot\bar{a}_\varphi = 4(\bar{a}_x\cdot\bar{a}_\varphi) - 2(\bar{a}_y\cdot\bar{a}_\varphi) - 4(\bar{a}_z\cdot\bar{a}_\varphi)$

$\boxed{A_\varphi = -4\sin\varphi - 2\cos\varphi}$ — ②

$A_z = \bar{A}\cdot\bar{a}_z = 4(\bar{a}_x\cdot\bar{a}_z) - 2(\bar{a}_y\cdot\bar{a}_z) - 4(\bar{a}_z\cdot\bar{a}_z)$

$\boxed{A_z = -4}$ — ③

at $(2,3,5)$, $\varphi = \tan^{-1}(y/x) = \tan^{-1}(3/2)$

$\boxed{\varphi = 56.31^\circ}$ — ④

$\therefore$ from eqn ①, $A_r = 4\cos(56.31^\circ) - 2\sin(56.31^\circ)$

$$\boxed{A_r = 0.555}$$

$$A_\phi = -4\sin\phi - 2\cos\phi$$

$$= -4 \times \sin(56.31^\circ) - 2\cos(56.31^\circ)$$

$$\boxed{A_\phi = -4.44}$$

$$\therefore \bar{A} = A_r\,\bar{a}_r + A_\phi\,\bar{a}_\phi + A_z\,\bar{a}_z$$

$$\boxed{\bar{A} = 0.555\,\bar{a}_r - 4.44\,\bar{a}_\phi - 4\,\bar{a}_z}$$

Problem ③ Express vector $\bar{B}$ in cartesian and cylindrical system. Given $\bar{B} = \frac{10}{r}\bar{a}_r + r\cos\theta\,\bar{a}_\theta + \bar{a}_\phi$. Then find $\bar{B}$ at $(-3, 4, 0)$ and $(5, \pi/2, -2)$

Solution:

$$\bar{B} = \frac{10}{r}\bar{a}_r + r\cos\theta\,\bar{a}_\theta + \bar{a}_\phi$$

General $\bar{B} = B_r\,\bar{a}_r + B_\theta\,\bar{a}_\theta + B_\phi\,\bar{a}_\phi$

$$\therefore B_r = \frac{10}{r}, \quad B_\theta = r\cos\theta, \quad B_\phi = 1$$

We know that

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} B_r \\ B_\theta \\ B_\phi \end{bmatrix}$$

$$B_x = B_r\sin\theta\cos\phi + B_\theta\cos\theta\cos\phi - B_\phi\sin\phi$$

$$= \frac{10}{r}\sin\theta\cos\phi + r\cos\theta\cos\theta\cos\phi - 1.\sin\phi$$

$$B_y = B_r\sin\theta\sin\phi + B_\theta\cos\theta\sin\phi + B_\phi\cos\phi$$

$$= \frac{10}{r}\sin\theta\sin\phi + r\cos\theta\cos\theta\sin\phi + 1.\cos\phi$$

$$B_z = B_r\cos\theta - B_\theta\sin\theta + 0$$

$$= \frac{10}{r}\cos\theta - r\cos\theta\sin\theta$$

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But $r = \sqrt{x^2+y^2+z^2}$ $\qquad \sin\theta = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$

$\cos\theta = \frac{z}{\sqrt{x^2+y^2+z^2}}$

$\sin\varphi = \frac{y}{\sqrt{x^2+y^2}}$ ✓

$\tan\varphi = \frac{y}{x}$ ✓

$\cos\varphi = \frac{x}{\sqrt{x^2+y^2}}$ ✓

$\bar{B}$ in cartesian system is given by

$$\bar{B} = \bar{B}_x \cdot \bar{a}_x + \bar{B}_y \bar{a}_y + \bar{B}_z \cdot \bar{a}_z$$

$$B_x = \frac{10}{r} \sin\theta \cos\varphi + r\cos^2\theta \cos\varphi - \sin\varphi$$

$$= \frac{10}{\sqrt{x^2+y^2+z^2}} \times \frac{\cancel{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2+z^2}} \times \frac{x}{\cancel{\sqrt{x^2+y^2}}} + \sqrt{x^2+y^2+z^2} \times \left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)^2 \times \frac{x}{\sqrt{x^2+y^2}} - \frac{y}{\sqrt{x^2+y^2}}$$

$$B_x = \frac{10x}{x^2+y^2+z^2} + \frac{x z^2}{\sqrt{x^2+y^2+z^2} \times \sqrt{x^2+y^2}} - \frac{y}{\sqrt{x^2+y^2}}$$

At $(-3, 4, 0)$, ie $x = -3$, $y = 4$, $z = 0$.

$$\therefore B_x = \frac{10 \times -3}{-3^2+4^2} + \frac{-3 \times 0}{\sqrt{-3^2+4^2} \cdot \sqrt{-3^2}} - \frac{4}{\sqrt{-3^2+4^2}}$$

$$= \frac{-30}{25} + 0 - \frac{4}{5} = \frac{-50}{25}$$

$$\boxed{B_x = -2}$$

Similarly $B_y = \frac{10}{\sqrt{x^2+y^2+z^2}} \times \frac{\cancel{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2+z^2}} \times \frac{y}{\cancel{\sqrt{x^2+y^2}}} + \sqrt{x^2+y^2+z^2} \times$

$$\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)^2 \times \frac{y}{\sqrt{x^2+y^2}} + \frac{x}{\sqrt{x^2+y^2}}$$

$$B_y = \frac{10y}{x^2+y^2+z^2} + \frac{y z^2}{\sqrt{x^2+y^2+z^2} \times \sqrt{x^2+y^2}} + \frac{x}{\sqrt{x^2+y^2}}$$

At $(-3, 4, 0)$ ie $x = -3$, $y = 4$, $z = 0$

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$$B_y = \frac{10 \times 4}{-3^2 + 4^2} + \frac{4 \times 0}{\sqrt{25}\sqrt{25}} + \frac{-3}{\sqrt{-3^2+4^2}}$$

$$= \frac{40}{25} + 0 - \frac{3}{5} = \frac{40 - 15}{25}$$

$$\boxed{B_y = 1}$$

$$B_z = \frac{10}{r}\cos\theta - r\sin\theta\cos\theta$$

$$= \frac{10}{\sqrt{x^2+y^2+z^2}} \times \frac{z}{\sqrt{x^2+y^2+z^2}} - \sqrt{x^2+y^2+z^2} \times \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} \times \frac{z}{\sqrt{x^2+y^2+z^2}}$$

At $(-3, 4, 0)$, ie $x = -3$, $y = 4$, $z = 0$

$$\therefore \boxed{B_z = 0}$$

In cartesian form,

$$\boxed{\bar{B} = -2\,\bar{a}_x + \bar{a}_y}$$

For transforming spherical to cylinderical,

$$\begin{vmatrix} B_\rho \\ B_\phi \\ B_z \end{vmatrix} = \begin{vmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{vmatrix} \begin{vmatrix} B_r \\ B_\theta \\ B_\phi \end{vmatrix}$$

$$B_\rho = B_r \sin\theta + B_\theta \cos\theta + 0$$

$$= \frac{10}{r}\sin\theta + r\cos^2\theta$$

at point $(5, \pi/2, -2)$

$$B_\phi = B_\phi = 1$$

$$B_z = \cos\theta\, B_r - \sin\theta\, B_\theta + 0$$

$$= \frac{10\cos\theta}{r} - r\sin\theta\cos\theta$$

$\rho = r\sin\theta$, $z = r\cos\theta$, $\varphi = \varphi$

$r = \sqrt{\rho^2 + z^2}$ $\theta = \tan^{-1}\left(\frac{\rho}{z}\right)$

$\tan\theta = \frac{\rho}{z}$, $\sin\theta = \frac{\rho}{\sqrt{\rho^2+z^2}}$, $\cos\theta = \frac{z}{\sqrt{(\rho^2+z^2)}}$

then ,

$\bar{B} = \bar{B}_\rho \bar{a}_\rho + \bar{B}_\varphi \bar{a}_\varphi + \bar{B}_z \bar{a}_z$

where $B_\rho = \frac{10\,\sin\theta}{r} - r\cos^2\theta$.

$= \frac{10 \times \frac{\rho}{\sqrt{\rho^2+z^2}}}{\sqrt{\rho^2+z^2}} - \sqrt{\rho^2+z^2} \times \left(\frac{z}{\sqrt{\rho^2+z^2}}\right)^2$

$B_\rho = \frac{10\rho}{\rho^2+z^2} + \frac{z^2}{\sqrt{\rho^2+z^2}}$

At given point $(5, \pi/2, -2)$, $\rho = 5$, $\varphi = \pi/2$, $z = -2$

$\therefore B_\rho = \frac{10 \times 5}{5^2+(-2)^2} + \frac{(-2)^2}{\sqrt{5^2+(-2)^2}}$

$= \frac{50}{29} + \frac{4}{\sqrt{29}}$

$\boxed{B_\rho = 2.467}$

$\boxed{B_\varphi = 1}$

$B_z = \frac{10\cos\theta}{r} - r\sin\theta\cos\theta$.

$= \frac{10}{\sqrt{\rho^2+z^2}} \times \frac{z}{\sqrt{\rho^2+z^2}} - \sqrt{\rho^2+z^2} \times \frac{\rho}{\sqrt{\rho^2+z^2}} \times \frac{z}{\sqrt{\rho^2+z^2}}$

$B_z = \frac{10z}{\rho^2+z^2} - \frac{\rho z}{\sqrt{\rho^2+z^2}}$

At given point $(5, \pi/2, -2)$, $\rho = 5$, $\varphi = \pi/2$, $z = -2$

$B_z = \frac{10 \times -2}{25+4} - \frac{5 \times -2}{\sqrt{25+4}}$

$$\boxed{B_z = 1.167}$$

In cylinderical form

$$\bar{B} = 2.467\,\overline{a_\rho} + \overline{a_\varphi} + 1.167\,\overline{a_z}$$

Question (4) If $B = y\,\bar{a}_x + (x+z)\,a_y$ and a point Q is located at $(-2, 6, 3)$ express 1) The point Q in cylinderical and spherical coordinates, 2) Spherical co ordinates:

Solution: $Q = (-2, 6, 3)$ ie $x = -2$, $y = 6$, $z = 3$

1) cylinderical, $r = \sqrt{x^2+y^2}$

$$r = \sqrt{(-2)^2 + 6^2} = \sqrt{4+36}$$

$$\boxed{r = 6.3245}$$

$$\varphi = \tan^{-1}(y/x) = \tan^{-1}(6/-2)$$

$$\boxed{\varphi = -71.565^\circ}$$

But x is negative φ must be in second quadrant hence add 180°

$$\therefore \varphi = -71.565 + 180$$

$$\boxed{\varphi = 108.435^\circ}$$

$$\boxed{z = 3}$$

$\therefore$ Q at cylinder $(6.3245, 108.435, 3)$

In spherical, $r = \sqrt{x^2+y^2+z^2} = \sqrt{(-2)^2+6^2+3^2}$

$$\boxed{r = 7}$$

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$\theta = \cos^{-1}\left[\frac{z}{r}\right] = 64.623^\circ$

$\phi = 108.435^\circ$

$Q(7, 64.623, 108.435^\circ)$

2) $\overline{B}$ in spherical coordinates.

$\overline{B} = \overline{B_r}\,\overline{a_r} + B_\theta \overline{a_\theta} + \overline{B_\phi}\,\overline{a_\phi}$

$\overline{B_r} = \overline{B} \cdot \overline{a_r}$

$= y(\overline{a_x} \cdot \overline{a_r}) + (x+z)(\overline{a_y} \cdot \overline{a_r})$

$B_r = y \sin\theta \cos\phi + (x+z) \sin\theta \sin\phi$

But- $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$,

$z = r\cos\theta$

$\therefore B_r = r\sin\theta\sin\phi\sin\theta\cos\phi + (r\sin\theta\cos\phi +$
$r\cos\theta)\sin\theta\sin\phi$

$= r\sin^2\theta\sin\phi\cos\phi + r\sin^2\theta\sin\phi\cos\phi$
$+ r\sin\theta\sin\phi\cos\theta$

$$\boxed{B_r = 2r\sin^2\theta\sin\phi\cos\phi + r\sin\theta\sin\phi\cos\theta}$$

similarly, $B_\theta = \overline{B} \cdot \overline{a_\theta} = y(\overline{a_x} \cdot \overline{a_\theta}) + (x+z)(\overline{a_y} \cdot \overline{a_\theta})$

$= y\cos\theta\cos\phi + (x+z)\cos\theta\sin\phi$

$$\boxed{B_\theta = 2r\sin\theta\cos\theta\sin\phi\cos\phi + r\cos^2\theta\sin\phi}$$

$B_\phi = \overline{B} \cdot \overline{a_\phi} = y(\overline{a_x} \cdot \overline{a_\phi}) + (x+z)(\overline{a_y} \cdot \overline{a_\phi})$

$= y(-\sin\phi) + (x+z)\cos\phi$

$B_\phi = -r\sin\theta\sin^2\phi + r\sin\theta\cos^2\phi + r\cos\theta\cos\phi$

$$\boxed{B_\phi = r\sin\theta\cos 2\theta + r\cos\theta\cos\phi}$$

$\therefore \quad \bar{B} = B_r \bar{a}_r + B_\theta \bar{a}_\theta + B_\varphi \bar{a}_\varphi$

At $Q \left\{ r = 7, \theta = 64.623, \varphi = 108.435 \right.$

$\bar{B} = -0.8571 \times \bar{a}_r - 0.4064 \bar{a}_\theta - 6 \bar{a}_\varphi$

Divergence Theorem:-

The integral of the normal component of any vector field over a closed surface is equal to the integral of the divergence of this vector field throughout the volume enclosed by the closed surface.

$$\nabla \cdot \bar{F} = \lim_{\Delta v \to 0} \frac{\oint \bar{F} \cdot \bar{ds}}{\Delta v}$$

and $\oint \bar{F} \cdot \bar{ds} = \int_V (\nabla \cdot \bar{F})\, dv.$

Problem ⑤, Using Divergence theorem, evaluate

$\int_S \bar{A} \cdot \bar{ds}$ where $\bar{A} = 2xy\, \bar{a}_x + y^2 \bar{a}_y + 4yz\, \bar{a}_z$

and s is the surface of the cube bounded by

$x=0, x=1, y=0, y=1, z=0, z=1$

Solution:- $\bar{A} = 2xy\, \bar{a}_x + y^2 \bar{a}_y + 4yz\, \bar{a}_z$

using divergence theorem, $\oint \bar{A} \cdot \bar{ds} = \oint_V (\nabla \cdot \bar{A})\, dv$

$$\nabla \cdot \bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$= 2y + 2y + 4y = 8y$$

$$\oint \bar{A} \cdot \bar{ds} = \oint_V 8y \, dv$$

$$= \int_{z=0}^{1} \int_{y=0}^{1} \int_{x=0}^{1} 8y \, dx \, dy \, dz$$

$$= 8 \left[\frac{y^2}{2} \right]_0^1 [x]_0^1 [z]_0^1$$

$$\oint \bar{A} \cdot ds = 8 \times \frac{1}{2} = 4$$

Gradient of a Scalar:-

consider vector operator in cartesian system denoted as ∇ called del, It defineds as

$$\nabla \text{ (del)} = \frac{\partial}{\partial x} \bar{a_x} + \frac{\partial}{\partial y} \bar{a_y} + \frac{\partial}{\partial z} \cdot \bar{a_z}$$

Ex

$$\text{Grad } W = \nabla W = \left[\frac{\partial}{\partial x} \bar{a_x} + \frac{\partial}{\partial y} \cdot \bar{a_y} + \frac{\partial}{\partial z} \cdot \bar{a_z} \right] W$$

co ordinate system	Grad W = ∇W
cartesian	$\nabla W = \frac{\partial W}{\partial x} \bar{a_x} + \frac{\partial W}{\partial y} \bar{a_y} + \frac{\partial W}{\partial z} \cdot \bar{a_z}$
cylinderical	$\nabla W = \frac{\partial W}{\partial r} \bar{a_r} + \frac{1}{r} \frac{\partial W}{\partial \phi} \cdot \bar{a_\phi} + \frac{\partial W}{\partial z} \bar{a_z}$
Spherical	$\nabla W = \frac{\partial W}{\partial r} \cdot \bar{a_r} + \frac{1}{r} \frac{\partial W}{\partial \theta} \bar{a_\theta} + \frac{1}{r \sin\theta} \frac{\partial W}{\partial \phi} \bar{a_\phi}$

Properties of gradient of scalar (∇W)

* It gives maximum rate of change of W per unit distance

* It always indicates the direction of the maximum rate of change of W.

* ∇W at any point is perpendicular to the constant W surface, which passes through the point.

* The directional derivate of W along the unit vector $\bar{a}$ is $\nabla W \cdot \bar{a}$ (dot product). which is projection of ∇W in the direction of unit vector. $\bar{a}$.

* Let W & u is the scalar function, then

$$\nabla (u+v) = \nabla U + \nabla W$$

$$\nabla (u v) = u \nabla W + W \nabla U$$

$$\nabla \left(\frac{u}{v}\right) = \frac{W \nabla u - U \nabla W}{W^2}$$

x — x — x

Question :6

Find the gradient of Scalar System $t = x^2 y + e^z$ at point P(1, 5, -2)

Solution:

$t = x^2 y + e^z$ & P(1, 5, -2)

$$\text{Gradient } t = \nabla t = \frac{\partial t}{\partial x}\bar{a_x} + \frac{\partial t}{\partial y}\cdot\bar{a_y} + \frac{\partial t}{\partial z}\bar{a_z}$$

$$\nabla t = 2xy\,\bar{a_x} + x^2\bar{a_y} + e^z\bar{a_z}$$

At $P(1,5,-2)$ i.e. $x=1$, $y=5$, $z=-2$

$$\boxed{\nabla t = 10\,\bar{a_x} + \bar{a_y} + e^{-2}\bar{a_z}}$$

x x

Problem no: 7

Find the gradient of following scalar fields

1) $V = e^{-z}\sin 2x \cosh y$ 2) $U = \rho^2 z \cos 2\phi$

3) $W = 10\,r \sin^2\theta \cos\phi$

Solution :-

1) $V = e^{-z}\sin 2x \cosh y$

$$\nabla V = \frac{\partial V}{\partial x}\bar{a_x} + \frac{\partial V}{\partial y}\bar{a_y} + \frac{\partial V}{\partial z}\bar{a_z}$$

$$\frac{\partial V}{\partial x} = \frac{\partial}{\partial x}\left[e^{-z}\sin 2x \cosh y\right] = 2e^{-z}\cos 2x \cosh y$$

$$\frac{\partial V}{\partial y} = \frac{\partial}{\partial y}\left[e^{-z}\sin 2x \cosh y\right] = e^{-z}\sin 2x \sinh y$$

$$\frac{\partial V}{\partial z} = \frac{\partial}{\partial z}\left[e^{-z}\sin 2x \cosh y\right] = -e^{-z}\sin 2x \cosh y.$$

$$\therefore \nabla V = 2e^{-z}\cos 2x \cosh y + e^{-z}\sin 2x \sinh y + -e^{-z}\sin 2x \cosh y.$$

2) $U = \rho^2 z \cos 2\phi$

$$\nabla U = \frac{\partial u}{\partial \rho}\bar{a_\rho} + \frac{1}{\rho}\frac{\partial u}{\partial \phi}\bar{a_\phi} + \frac{\partial u}{\partial z}\bar{a_z}$$

$$= 2\rho z\cos 2\phi\,\bar{a_\rho} + \frac{1}{\rho}\rho^2 z(-\sin 2\phi)\times 2\,\bar{a_\phi} + \rho^2\cos 2\phi\,\bar{a_z}$$

$$= 2\rho z\cos 2\phi\,\bar{a_\rho} - 2\rho z\sin 2\phi\,\bar{a_\phi} + \rho^2\cos 2\phi\,\bar{a_z}$$

3) $W = 10 r \sin^2\theta \cos\varphi$

$$\nabla W = \frac{\partial W}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial W}{\partial \theta}\bar{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial W}{\partial \phi}\bar{a}_\varphi$$

$$= 10\sin^2\theta\cos\varphi\,\bar{a}_r + \frac{1}{r}\times 10r\cos\varphi \times 2\sin\theta\cos\theta\,\bar{a}_\theta + \frac{1}{r\sin\theta}\times 10 r \sin^{\cancel{2}}\theta\,(-\sin\varphi)\,\bar{a}_\varphi$$

$$\nabla W = 10\sin^2\theta\cos\varphi\,\bar{a}_r + 10\cos\varphi\sin 2\theta\,\bar{a}_\theta - 10\sin\theta\sin\varphi\,\bar{a}_\varphi$$

Curl of a Vector:

$$\text{Curl of } \bar{F} = \lim_{\Delta S_N \to 0} \frac{\oint F\cdot \bar{dl}}{\Delta S_N}$$

$\Delta S_N \Rightarrow$ area enclosed by the line integral in normal direction.

$$\boxed{\nabla \times \bar{F} = \text{Curl of } \bar{F}}$$

curl indicates the rotational property of vector field. If curl of vector is zero, the vector field is irrotational

$$\boxed{\nabla \times \vec{F} = 0 \quad \text{then irrotational}}$$

$$\nabla\times\vec{F} = \left[\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right]\bar{a}_x + \left[\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right]\bar{a}_y + \left[\frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x}\right]\bar{a}_z$$

$$\nabla \times \bar{F} = \begin{bmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{bmatrix}$$

→ cartesian coordinates

$$\nabla \times \bar{F} = \frac{1}{r}\left[\frac{\partial F_z}{\partial \varphi} - \frac{\partial F_\varphi}{\partial z}\right]\bar{a}_r + \left[\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right]\bar{a}_\varphi +$$

$$\left[\frac{\partial F_r}{\partial \varphi} - \frac{\partial F_\varphi}{\partial r}\right]\bar{a}_z$$

$$\nabla \times \bar{F} = \frac{1}{r}\begin{bmatrix} \bar{a}_r & r\bar{a}_\varphi & \bar{a}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ F_r & rF_\varphi & F_z \end{bmatrix}$$

→ cylinderical.

$$\nabla \times \bar{F} = \frac{1}{r\sin\theta}\left[\frac{\partial F_\varphi \sin\theta}{\partial \theta} - \frac{\partial F_\theta}{\partial \varphi}\right]\bar{a}_r +$$

$$\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial F_r}{\partial \varphi} - \frac{\partial (rF_\varphi)}{\partial r}\right]\bar{a}_\theta +$$

$$\frac{1}{r}\left[\frac{\partial (rF_\theta)}{\partial r} - \frac{\partial F_r}{\partial \theta}\right]\bar{a}_\varphi$$

$$\nabla \times F = \frac{1}{r^2\sin\theta}\begin{bmatrix} \bar{a}_r & r\bar{a}_\theta & r\sin\theta\,\bar{a}_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ F_r & rF_\theta & r\sin\theta F_\varphi \end{bmatrix}$$

→ Spherical

---x---x---

Problem 8:

Determine the divergence and curl of the vector $\bar{A} = x^2\,\bar{a_x} + y^2\,\bar{a_y} + y^2\,\bar{a_z}$

Solution:

$$A = x^2\,\bar{a_x} + y^2\,\bar{a_y} + y^2\,\bar{a_z}$$

$$\nabla \cdot \bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$= 2x + 2y + 0$$

$$\boxed{\nabla \cdot \bar{A} = 2(x+y)}$$

$$\nabla \times \bar{A} = \begin{bmatrix} \bar{a_x} & \bar{a_y} & \bar{a_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & y^2 \end{bmatrix}$$

$$= \bar{a_x}\left[\frac{\partial y^2}{\partial y} - \frac{\partial y^2}{\partial z}\right] - \bar{a_y}\left[\frac{\partial y^2}{\partial x} - \frac{\partial x^2}{\partial z}\right] + \bar{a_z}\left[\frac{\partial y^2}{\partial x} - \frac{\partial x^2}{\partial y}\right]$$

$$= \bar{a_x}\,[2y - 0] + \bar{a_y}\,[0-0] + \bar{a_z}\,[0-0]$$

$$\boxed{\nabla \times \bar{A} = 2y\,\bar{a_x}}$$

Determine the curl of these vector fields:

(i) $\bar{P} = x^2yz\,\bar{a_x} + xz\,\bar{a_z}$

(2) $\bar{Q} = \rho \sin\phi\,\bar{a_\rho} + \rho^2 z\,\bar{a_\varphi} + z\cos\varphi\,\bar{a_z}$

(3) $T = \frac{1}{r^2}\cos\theta\,\bar{a_r} + r\sin\theta\cos\varphi\,\bar{a_\theta} + \cos\theta\,\bar{a}$

Solution:

(i) $\bar{P} = \underbrace{x^2yz}_{P_x}\,\bar{a_x} + \underbrace{xz}_{P_z}\,\bar{a_z}$ $\quad P_y = 0$

$$\nabla \times \bar{P} = \left[\frac{\partial P_z}{\partial y} - \frac{\partial P_y}{\partial z}\right]\bar{a_x} + \left[\frac{\partial P_x}{\partial z} - \frac{\partial P_z}{\partial x}\right]\bar{a_y} + \left[\frac{\partial P_x}{\partial y} - \frac{\partial P_y}{\partial x}\right]\bar{a_z}$$

$$= [0-0]\,\bar{a_x} + [x^2y - z]\,\bar{a_y} + [0 - x^2 z]\,\bar{a_z}$$

$$\nabla \times \bar{P} = (x^2y - z)\,\bar{a_y} - x^2 z\,\bar{a_z}$$

(2) $\bar{Q} = \underbrace{\rho \sin\varphi}_{Q_\rho}\,\bar{a_\rho} + \underbrace{\rho^2 z}_{Q_\varphi}\,\bar{a_\varphi} + \underbrace{z\cos\varphi}_{Q_z}\,\bar{a_z}$

$$\nabla \times \bar{Q} = \left[\frac{1}{\rho}\frac{\partial Q_z}{\partial \varphi} - \frac{\partial Q_\varphi}{\partial z}\right]\bar{a_\rho} + \left[\frac{\partial Q_\rho}{\partial z} - \frac{\partial Q_z}{\partial \rho}\right]\bar{a_\varphi} + \left[\frac{1}{\rho}\frac{\partial \rho Q_\varphi}{\partial \varphi} - \frac{1}{\rho}\frac{\partial Q_\rho}{\partial \varphi}\right]\bar{a_z}$$

$$= \left[\frac{1}{\rho}\, z \cdot - z\sin\varphi - \rho^2\right]\bar{a_\rho} + [0-0]\,\bar{a_\varphi} + \left[\frac{1}{\rho}\, 3\rho^2 z - \frac{1}{\rho}\,\rho\cos\varphi\right]\bar{a_z}$$

$$\boxed{\nabla \times \bar{Q} = \left(-\frac{z\sin\varphi}{\rho} - \rho^2\right)\bar{a_\rho} + (3\rho z - \cos\varphi)\,\bar{a_z}}$$

(3)

$$\nabla \times \bar{P} = \left[\frac{\partial P_z}{\partial y} - \frac{\partial P_y}{\partial z}\right]\bar{a_x} + \left[\frac{\partial P_x}{\partial z} - \frac{\partial P_z}{\partial x}\right]\bar{a_y} + \left[\frac{\partial P_x}{\partial y} - \frac{\partial P_y}{\partial x}\right]\bar{a_z}$$

$$= [0-0]\,\bar{a_x} + [x^2y - z]\,\bar{a_y} + [0 - x^2 z]\,\bar{a_z}$$

$$\nabla \times \bar{P} = (x^2 y - z)\,\bar{a_y} - x^2 z\,\bar{a_z}$$

(2) $\bar{Q} = \underbrace{\rho \sin\varphi}_{Q_\rho}\,\bar{a_\rho} + \underbrace{\rho^2 z}_{Q_\phi}\,\bar{a_\varphi} + \underbrace{z\cos\varphi}_{Q_z}\,\bar{a_z}$

$$\nabla \times \bar{Q} = \left[\frac{1}{\rho}\frac{\partial Q_z}{\partial \phi} - \frac{\partial Q_\phi}{\partial z}\right]\bar{a_\rho} + \left[\frac{\partial Q_\rho}{\partial z} - \frac{\partial Q_z}{\partial \rho}\right]\bar{a_\varphi} + \left[\frac{1}{\rho}\frac{\partial \rho Q_\phi}{\partial \rho} - \frac{1}{\rho}\frac{\partial Q_\rho}{\partial \varphi}\right]\bar{a_z}$$

$$= \left[\frac{1}{\rho} \times -z\sin\phi - \rho^2\right]\bar{a_\rho} + [0-0]\,\bar{a_\varphi} + \left[\frac{1}{\rho}3\rho^2 z - \frac{1}{\rho}\rho\cos\phi\right]\bar{a_z}$$

$$\boxed{\nabla \times \bar{Q} = \left(-\frac{z\sin\varphi}{\rho} - \rho^2\right)\bar{a_\rho} + (3\rho z - \cos\varphi)\,\bar{a_z}}$$

(3) $T = \frac{1}{r^2}\cos\theta\,\bar{a_r} + r\sin\theta\cos\varphi\,\bar{a_\theta} + \cos\theta\,\bar{a_\varphi}$

$T_r = \frac{1}{r^2}\cos\theta$, $T_\theta = r\sin\theta\cos\phi$, $T_\varphi = \cos\theta$

$$\nabla \times T = \frac{1}{r\sin\theta}\left[\frac{\partial T_\varphi \sin\theta}{\partial \theta} - \frac{\partial T_\theta}{\partial \varphi}\right]\bar{a_r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial T_r}{\partial \varphi} - \frac{\partial (r T_\phi)}{\partial r}\right]\bar{a_\theta} + \frac{1}{r}\left[\frac{\partial (r T_\theta)}{\partial r} - \frac{\partial T_r}{\partial \theta}\right]\bar{a_\phi}$$

$$= \frac{1}{r\sin\theta}\left[\frac{\partial(\sin\theta\cos\theta)}{\partial\theta} - r\sin\theta(-\sin\varphi)\right]\bar{a_r} + \frac{1}{r}\left[\frac{1}{\sin\theta}(0) - \frac{\partial(r\cos\theta)}{\partial r}\right]\bar{a_\theta} + \frac{1}{r}\left[\frac{\partial(r^2\sin\theta\cos\varphi)}{\partial r} - \frac{1}{r^2}(-\sin\theta)\right]\vec{a_\phi}$$

$$\Rightarrow \frac{\partial(\sin\theta\cos\theta)}{\partial\theta} = \frac{1}{2}\frac{\partial(2\sin\theta\cos\theta)}{\partial\theta}$$

$$= \frac{1}{2}\frac{\partial(\sin 2\theta)}{\partial\theta}$$

$$= \frac{1}{2}\times 2\cos 2\theta = \cos 2\theta .$$

$$\nabla\times T = \frac{1}{r\sin\theta}\left[\cos 2\theta + r\sin\theta\sin\varphi\right]\bar{a}_r + \frac{1}{r}\left[-\cos\theta\right]\bar{a}_\theta + \frac{1}{r}\left[2r\sin\theta\cos\varphi + \frac{\sin\theta}{r^2}\right]\bar{a}_\varphi$$

$$\nabla\times T = \left[\frac{\cos 2\theta}{r\sin\theta} + \sin\varphi\right]\bar{a}_r - \frac{\cos\theta}{r}\bar{a}_\theta + \left[2\sin\theta\cos\varphi + \frac{\sin\theta}{r^3}\right]\bar{a}_\varphi$$

Stoke's Theorem

Stoke's Theorem relates the line integral to surface integral. It states that

"The line integral of $\bar{F}$ around a closed path L is equal to the integral of curl of $\bar{F}$ over the open surface S enclosed by the closed path L."

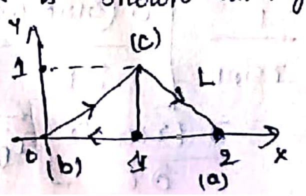
$$\oint_L \bar{F}\cdot \bar{dL} = \int_S (\nabla\times\bar{F})\cdot ds$$

$dL \rightarrow$ perimeter of total surface.

Problem ④ Given that $\bar{F} = x^2 y\, \bar{a}_x - y\, \bar{a}_y$

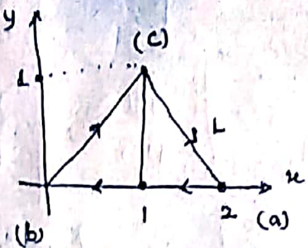
(i) Find $\oint_L \bar{F}\cdot\bar{dL}$ where L is shown in Fig

(ii) Verify Stoke's Theorem.



Solution (i) The path of L is shown.

$$\oint_L \bar{F}\cdot d\bar{L} = \int_{ab} + \int_{bc} + \int_{ca} \bar{F}\cdot d\bar{L}$$



$$\oint_{ab} \bar{F}\cdot d\bar{l} = \int_{ab} (x^2 y\, \bar{a_x} - y\, \bar{a_y}) \cdot dx\, \bar{a_x}$$

$$= \int_{x=2}^{0} x^2 y\, dx \quad \text{and } y=0 \text{ for path } ab$$

$$\oint_{ab} \bar{F}\cdot d\bar{L} = 0$$

$$\int_{bc} \bar{F}\cdot d\bar{L} = \int_{bc} (x^2 y\, \bar{a_x} - y\, \bar{a_y}) \cdot (dx\, \bar{a_x} + dy\, \bar{a_y})$$

$$= \int_{bc} x^2 y\, dx - y\, dy$$

equation of path bc is $y = x$, ie $dy = dx$.

$$\oint_{bc} \bar{F}\cdot d\bar{L} = \int_{x=0}^{1} x^3 dx - \int_{y=0}^{1} y\, dy$$

$$= \left.\frac{x^4}{4}\right|_0^1 - \left.\frac{y^2}{2}\right|_0^1$$

$$= \frac{1}{4} - \frac{1}{2}$$

$$\oint_{bc} \bar{F}\cdot d\bar{L} = -\frac{1}{4}$$

$$\oint_{ca} \bar{F}\cdot d\bar{L} = \oint_{ca} (x^2 y\, \bar{a_x} - y\, \bar{a_y}) \cdot [dx\, \bar{a_x} + dy\, \bar{a_y}]$$

$$= \int_{ca} x^2 y\, dx - y\, dy$$

equation of path ca is $y = mx + c$ where $m = -1$

slope and $y = 1$ for $x = 1$

$1 = -1 + c$ ie $c = 2$, $y = -x + 2$

$$\int_{ca} x^2 y\, dx - y\, dy = \int_{x=1}^{x=2} x^2 (-x+2)\, dx - \int_{y=1}^{0} y\, dy.$$

$$= -\frac{x^4}{4} + \frac{2x^3}{3}\Big]_1^2 - \frac{y^2}{2}\Big]_1^0$$

$$= -4 + \frac{1}{4} + \frac{16}{3} - \frac{2}{3} + \frac{1}{2}$$

$$= 1.4166$$

$$\oint_L \bar{F}.d\bar{l} = 0 - \frac{1}{4} + 1.4166 = 1.1667$$

(ii) For stokes theorem, find $\int_S (\nabla \times \bar{F}) \cdot \bar{ds}$

$$\nabla \times \bar{F} = \begin{bmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & -y & 0 \end{bmatrix} = \bar{a_x}(0-0) - \bar{a_y}(0-0) + \bar{a_z}(0 - x^2)$$

$$= -x^2 \bar{a_z}$$

$$\oint (\nabla \times \bar{F}).\bar{ds} = \int_S (-x^2 \bar{a_z}) \cdot dx\, dy\, \bar{a_z}$$

$$= \int_S -x^2 dx\, dy$$

Now the area will be splited. (two triangles).

1) $dy = x$ (x varies from 1 to 0)

2) $y = -x+2$, $dy = -x+2$, (x varies from 2 to 1)

$$\oint_S (\nabla \times \bar{F}).\bar{ds} = \int_{x=1}^{0} -x^2 \times x\, dx + \int_{x=2}^{1} -x^2(-x+2)dx$$

$$= \int_{x=1}^{0} -x^3 dx + \int_{x=2}^{1} (x^3 - 2x^2) dx$$

$$= \left[\frac{-x^4}{4}\right]_1^0 + \left[\frac{x^4}{4}\right]_2^1 - \left[\frac{2x^3}{3}\right]_2^1$$

$$= \frac{14}{12} = 1.667$$

Now stokes theorem is varified.

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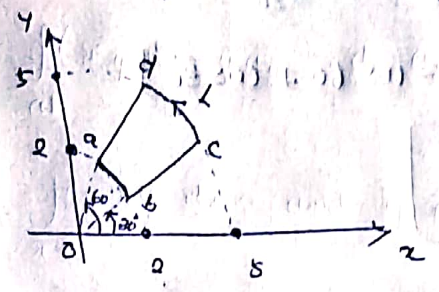
Problem (10) If $\bar{A} = \rho \cos\phi \, \bar{a}_\rho + \sin\phi \, \bar{a}_\phi$, evaluate $\oint \bar{A} \cdot dL$ around the path shown in fig. Confirm this using Stokes theorem.



Solution:

$$\oint_L \bar{A} \cdot \bar{dL} = \int_{ab} + \int_{bc} + \int_{cd} + \int_{da}$$

$\bar{dL} = d\rho \, \bar{a}_\rho + \rho \, d\phi \, \bar{a}_\phi + dz \, \bar{a}_z$ — Cylinderical system.

$\vec{A} = \rho \cos\phi \, \vec{a}_\rho + \sin\phi \, \bar{a}_\phi$

$A_\rho = \rho \cos\phi$, $A_\phi = \sin\phi$, $A_z = 0$

For path ab, the direction is a_ϕ hence ~~...~~

$$A \cdot dL = (\sin\phi)(\rho \, d\phi)$$

$$\int_{ab} \bar{A} \cdot \bar{dL} = \int_{\phi=60°}^{30°} \sin\phi \, \rho \, d\phi \quad \text{with } \rho = 2$$

$$= \left[-\cos\phi \right]_{60°}^{30} \times 2 = 2 \left[-0.866 + 0.5 \right]$$

$$\boxed{\int_{ab} \bar{A} \cdot \bar{dL} = -0.732}$$

For path bc, the direction is $\bar{a}_\rho$, hence

$$\boxed{\bar{A} \cdot \bar{dL} = \rho \cos\phi \, d\rho}$$

$$\int_{bc} \bar{A} \cdot \bar{dL} = \int_{\rho=2}^{5} \rho \cos\phi \, d\rho \quad \text{with } \phi = 30°$$

$$= \cos 30° \left[\frac{\rho^2}{2} \right]_2^5 = 0.866 \times \frac{25-4}{2}$$

$$\boxed{\int_{bc} \bar{A} \cdot \bar{dL} = 9.093}$$

For path cd, the direction is $\bar{a}_\phi$, $\bar{A} \cdot \bar{dL} = (\sin\phi)(\rho \, d\phi)$

$$\int_{cd} \bar{A}\cdot dL = \int_{\varphi=30^\circ}^{60^\circ} \sin\varphi \; \rho \, d\varphi \quad \text{with } \rho = 5$$

$$= 5\left[-\cos\varphi\right]_{30^\circ}^{60^\circ}$$

$$= 5\left[-\cos 60^\circ + \cos 30^\circ\right]$$

$$\boxed{\int_{cd} \bar{A}\cdot dL = 1.83}$$

For path da, the direction is $\bar{a}_\rho$, hence

$$\bar{A}\cdot dL = \rho \cos\varphi \, d\rho$$

$$\int_{da} \bar{A}\cdot d\bar{L} = \int_{\rho=5}^{2} \rho \cos\varphi \, d\rho \quad \text{with } \varphi = 60^\circ$$

$$= \cos 60^\circ \times \frac{\rho^2}{2}\Big|_5^2 = 0.5\left[\frac{4-25}{2}\right]$$

$$\boxed{\int_{da} \bar{A}\cdot d\bar{L} = -5.25}$$

$\therefore$

$$\int \bar{A}\cdot d\bar{L} = -0.732 + 9.093 + 1.83 - 5.25$$

$$= 4.941$$

For Stoke's theorem.

$$\nabla\times\bar{A} = \frac{1}{\rho}\left[\frac{\partial A_z}{\partial\varphi} - \frac{\partial A_\varphi}{\partial z}\right]\bar{a}_\rho + \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right]\bar{a}_\varphi$$

$$+\left[\frac{1}{\rho}\frac{\partial(\rho A_\varphi)}{\partial\rho} - \frac{1}{\rho}\frac{\partial A_\rho}{\partial\varphi}\right]a_z$$

$$= [0-0]\bar{a}_\rho + [0-0]\bar{a}_\varphi + \left[\frac{1}{\rho}\sin\varphi - \frac{1}{\rho}\rho(-\sin\varphi)\right]\bar{a}_z$$

$$= \sin\varphi\left[\frac{1+\rho}{\rho}\right]\bar{a}_z$$

$d\bar{s} = \rho \, d\rho \, d\varphi \; \bar{a}_z$ As surface is in x-y plane.

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$$\nabla \times \bar{A} \cdot \bar{ds} = \sin\phi \left(\frac{1+\rho}{\rho}\right) \rho \, d\rho \, d\phi$$

$$= \sin\phi \, (1+\rho) \, d\rho \, d\phi$$

$$\oint (\nabla \times \bar{A}) = \int_{\phi=30}^{60^\circ} \int_{\rho=2}^{5} \sin\phi \, (1+\rho) \, d\rho \, d\phi$$

$$= \left[-\cos\phi\right]_{30}^{60} \times \left[\rho + \frac{\rho^2}{2}\right]_2^5$$

$$= \left[-\cos 60^\circ + \cos 30^\circ\right] \left[5 + \frac{25}{2} - 2 - \frac{4}{2}\right]$$

$$\oint (\nabla \times \bar{A}) = 4.941$$

hence stokes theorem is verfied.

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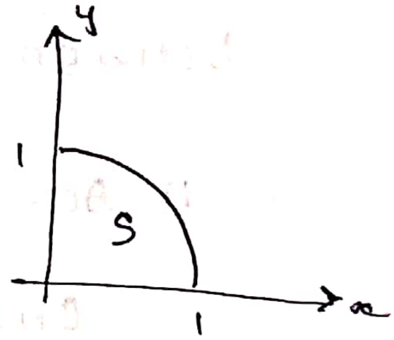
Problem (11) Given $\bar{A} = \rho \cos\varphi \, \bar{a}_\rho + \rho^2 a_z$

Compute $\nabla \times \bar{A}$ and $\int_S \nabla \times \bar{A} \cdot \bar{ds}$ over the area S as shown in fig



Solution:- $\bar{A} = \rho \cos\varphi \, \bar{a}_\rho + \rho^2 \bar{a}_z$

In cylindrical system,

$$\nabla \times \bar{A} = \left[\frac{1}{\rho}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right]\bar{a}_\rho + \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right]\bar{a}_\varphi$$

$$+ \left[\frac{1}{\rho}\frac{\partial(\rho A_\varphi)}{\partial \rho} - \frac{1}{\rho}\frac{\partial A_\rho}{\partial \varphi}\right]\bar{a}_z$$

Given: $A_\rho = \rho \cos\varphi$, $A_\varphi = 0$, $A_z = \rho^2$

$$\nabla \times \bar{A} = [0 - 0]\,\bar{a}_\rho + (0 - 2\rho)\,\bar{a}_\varphi + \left[0 - \frac{1}{\rho}(-\sin\varphi)\rho\right]\bar{a}_z$$

$$= -2\rho \, \bar{a}_\varphi + \sin\varphi \, \bar{a}_z$$

As the surface in x-y plane, $\bar{ds} = \rho \, d\rho \, d\varphi \, \bar{a}_z$

$$\oint_S (\nabla \times \bar{A}) \cdot \bar{ds} = \iint (\sin\varphi \, \bar{a}_z) \cdot [\rho \, d\rho \, d\varphi] \, \bar{a}_z$$

$\because \bar{a}_\varphi \cdot \bar{a}_z = 0$

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$$= \int_{\phi=0}^{\pi/2} \int_{\rho=0}^{1} \sin\phi \, \rho \, d\rho \, d\phi$$

$$= \left[-\cos\phi\right]_0^{\pi/2} \left[\frac{\rho^2}{2}\right]_0^1$$

$$= 0 + \left[-0 - (-1)\right]\left[\tfrac{1}{2}\right]$$

$$\int_S (\nabla \times \bar{A}) \, \bar{ds} = 1/2$$

Coulomb's law

The coulomb's law states that the force between the two point charges Q_1 and Q_2

1) Acts along the line joing the two point charges.

2) Is directly proportional to the product $(Q_1 Q_2)$. of the two charges.

3) Is inversly proportional to the square of the distance b/w them.

Force b/w the two charges,

$$F \propto \frac{Q_1 Q_2}{R^2}$$

$R \rightarrow$ Distance b/w the two charges.

$$F = K \cdot \frac{Q_1 Q_2}{R_2}$$

$K \rightarrow$ constant proportionality.

$$= \int_{\varphi=0}^{\pi/2} \int_{\rho=0}^{1} \sin\varphi \, \rho \, d\rho \, d\varphi$$

$$= \left[-\cos\varphi\right]_0^{\pi/2} \left[\frac{\rho^2}{2}\right]_0^1$$

$$= 0 + \left[-0-(-1)\right]\left[\tfrac{1}{2}\right]$$

$$\int_S (\nabla \times \bar{A}) \cdot \bar{ds} = 1/2$$

Coulomb's law

The coulomb's law states that the force between the two point charges Q_1 and Q_2

1) Acts along the line joing the two point charges.

2) Is directly proportional to the product $(Q_1 Q_2)$ of the two charges.

3) Is inversely proportional to the square of the distance b/w them.

Force b/w the two charges,

$$F \propto \frac{Q_1 Q_2}{R^2}$$

$R \rightarrow$ Distance b/w the two charges.

$$F = k \cdot \frac{Q_1 Q_2}{R_2}$$

$K \rightarrow$ constant proportionality.

$$K = \frac{1}{4\pi \varepsilon}$$

$\varepsilon \rightarrow$ permittivity of medium which charges are located. (F/m).

$$\varepsilon = \varepsilon_0 \varepsilon_r$$

$\varepsilon_0 \rightarrow$ permittivity of the free space or vacuum.

$\varepsilon_r \rightarrow$ relative permittivity or dielectric constant of the Medium with respect to free space.

$\varepsilon \rightarrow$ absolute permittivity.

For free space, permittivity $\varepsilon_r = 1$, hence

$$\varepsilon = \varepsilon_0$$

$$\therefore \quad F = \frac{1}{4\pi \varepsilon_0} \frac{Q_1 Q_2}{R^2}$$

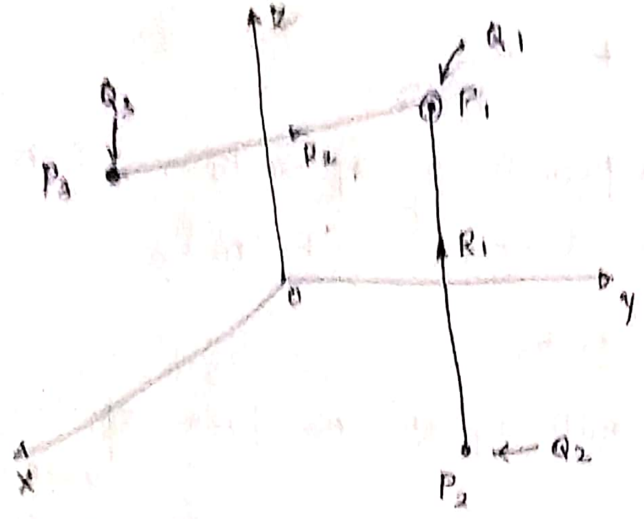
$$\varepsilon_0 = \frac{1}{36\pi} \times 10^{-9}$$

$$\boxed{\varepsilon_0 = 8.854 \times 10^{-12} \ F/m}$$

Coulomb's law can be expressed,

$$\boxed{F = \frac{Q_1 Q_2}{4\pi \varepsilon_0 R^2}}$$

Problem (12) A charge $Q_1 = 100 \ nC$ is located in Vacuum at $P_1(-0.03, 0.01, 0.04)$ m. Find the force on Q_1 due to (i) $Q_2 = 120 \ \mu C$ at $P_2(0.03, 0.08, -0.02)$ m (ii) $Q_3 = 120 \ \mu C$ at $P_3(-0.09, -0.06, 0.10)$ (iii) Q_2 and Q_3.



(i) <u>Force on Q_1 due to Q_2</u>

$Q_2 = 120\ \mu C \quad , P_2\ (0.03, 0.08, -0.02)$

$Q_1 = 100\ nC \quad , P_1\ (-0.03, 0.01, 0.04)$

$$\overline{R_1} = (-0.03 - 0.03)\,\overline{a_x} + (0.01 - 0.08)\,\overline{a_y} + (0.04 + 0.02)\,\overline{a_z}$$

$$\overline{R_1} = -0.06\,\overline{a_x} - 0.07\,\overline{a_y} + 0.06\,\overline{a_z}$$

Magnitude $|\overline{R_1}| = \sqrt{0.06^2 + 0.07^2 + 0.06^2}$

$$\boxed{|\overline{R_1}| = 0.11}$$

$$\overline{F_1} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_1^2}\,\overline{a_{R1}} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_1^2} \times \left[\frac{\overline{R_1}}{|R_1|}\right]$$

$$= \frac{120\times10^{-6} \times 100\times10^{-9}}{4\pi\times8.854\times10^{-12}\times0.11^2} \times \left[\frac{-0.06\,\overline{a_x} - 0.07\,\overline{a_y} + 0.06\,\overline{a_z}}{0.11}\right]$$

$$\boxed{\overline{F_1} \Rightarrow -4.862\,\overline{a_x} - 5.672\,\overline{a_y} + 4.862\,\overline{a_z} \quad N}$$

(ii) <u>Force On Q_1 due to Q_3</u>

$Q_3 = 120\ \mu c \qquad P_3 = (-0.09, 0.06, 0.10)$

$$\overline{R_2} = \left[-0.03 - (-0.09)\right]\overline{a_x} + \left[0.01 - (-0.06)\right]\overline{a_y} + \left[0.04 - 0.10\right]\overline{a_z}$$

$$\overline{R_2} = 0.06\,\overline{a_x} + 0.07\,\overline{a_y} - 0.06\,\overline{a_z}$$

$$|\overline{R_2}| = \sqrt{0.06^2 + 0.07^2 + 0.06^2}$$

$$\boxed{|\overline{R_2}| = 0.11}$$

$$\overline{F_2} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_2^2} \cdot \overline{a_{R_2}} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_2^2} \times \left[\frac{\overline{R_2}}{|\overline{R_2}|}\right]$$

$$= \frac{100\times10^{-9} \times 120\times10^{-6}}{4\pi\times8.854\times10^{-12}\times0.11^2} \times \left[\frac{0.06\,\overline{a_x} + 0.07\,\overline{a_y} - 0.06\,\overline{a_z}}{0.11}\right]$$

$$\boxed{\overline{F_2} = 4.862\,\overline{a_x} + 5.672\,\overline{a_y} - 4.862\,\overline{a_z}\ \text{N}}$$

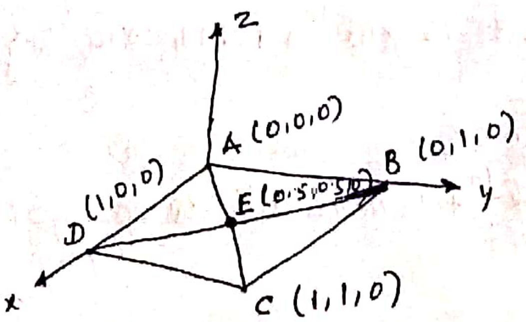
(iii) Force due to Q_2 and Q_3

$$F_t = \overline{F_1} + \overline{F_2} = 0\ \text{N}.$$

Problem (13) Four point charges of 10 μC each are placed at the corners of square of side 1 m Determine the value of charges that is to be placed at the centre of the square so that the system of charges is brought to equilibrium.

Solution The square is kept in x-y plane. The coordinates of various points are,

A (0,0,0) , B (0,1,0) , C (1,1,0), D (1,0,0),
E (0.5, 0.5, 0) .



At equilibrium, net force is zero.

Let Forces at A due to other charges,

$\bar{F_A} = \bar{F}$ due to charges at B, C, D & E.

The charge to be placed at E be 'Q' while charge at B, C, D is 10 μC each.

$$\therefore \bar{F_A} = \frac{Q_A Q_B}{4\pi\varepsilon_0 R_{AB}^2}\bar{a}_{AB} + \frac{Q_A Q_C}{4\pi\varepsilon_0 R_{AC}^2}\bar{a}_{AC} + \frac{Q_A Q_D}{4\pi\varepsilon_0 R_{AD}^2}\bar{a}_{AD} + \frac{Q_A Q_E}{4\pi\varepsilon_0 R_{AE}^2}\bar{a}_{AE}$$

$\bar{R}_{AB} = \bar{a}_y$, $\bar{R}_{AC} = \bar{a}_x$, $\bar{R}_{AD} = \bar{a}_x + \bar{a}_y$, $\bar{R}_{AE} = 0.5\bar{a}_x + 0.5\bar{a}_y$

$\therefore |\bar{R}_{AB}| = 1$, $|\bar{R}_{AC}| = 1$, $|\bar{R}_{AD}| = \sqrt{2}$, $|\bar{R}_{AE}| = 0.7071$

$$\bar{a}_R = \frac{\bar{R}}{|R|}$$

$Q_A = Q_B = Q_C = Q_D = 10\ \mu C$, $Q_E = Q$

$$\therefore \bar{F_A} = \frac{1}{4\pi\varepsilon_0}\left\{\frac{Q_A Q_B}{R_{AB}^2}\frac{\bar{a}_y}{|R_{AB}|} + \frac{Q_B Q_C}{R_{AC}^2} \times \frac{\bar{a}_x}{|R_{AC}|} + \frac{Q_A Q_D}{|R_{AD}|^2}\cdot\frac{\bar{a}_x + \bar{a}_y}{|R_{AD}|} + \frac{Q_A Q_E}{R_{AE}^2} \times \frac{0.5\bar{a}_x + 0.5\bar{a}_y}{|R_{AE}|}\right\}$$

$$\bar{F} = \frac{1}{4\pi\varepsilon_0}\left[\frac{(10\times10^{-6})^2}{1^2} \times \frac{\bar{a}_y}{1} + \frac{(10\times10^{-6})^2}{1^2} \times \frac{\bar{a}_x}{1} + \frac{(10\times10^{-6})^2}{(\sqrt{2})^2} \times \left[\frac{\bar{a}_x + \bar{a}_y}{\sqrt{2}}\right] + \frac{10\times10^{-6}\times Q}{(0.7071)^2} \times \frac{0.5\bar{a}_x + 0.5\bar{a}_y}{0.707}\right]$$

$$\bar{F} = \frac{1}{4\pi\varepsilon_0}\left\{\{1.3535\times10^{-10} + 1.4142\times10^{-5}Q\}\bar{a}_x + \{1.3535\times10^{-10} + 1.4142\times10^{-5}Q\}\bar{a}_y\right\}$$

For equilibrium, $\bar{F}_A = 0$,

$$1.3535\times10^{-10} + 1.4142\times10^{-5}Q = 0$$

$$Q = \frac{-1.3535 \times 10^{-10}}{1.4142 \times 10^{-5}}$$

$$\boxed{\therefore Q = -9.5704 \ \mu C}$$

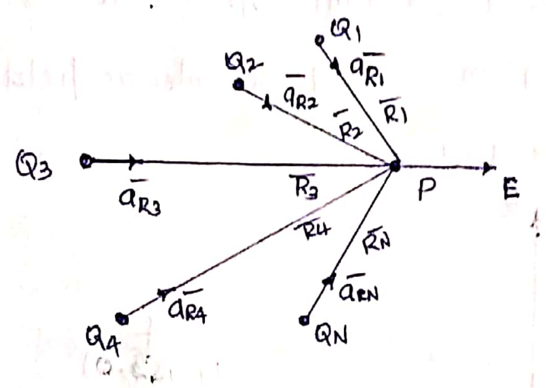
Electric Field Intensity

The force exerted by unit charge is called Electric field intensity (or) electric field strength.

$$\bar{E} = \frac{Q_1}{4\pi\varepsilon_0 R_{1p}^2} \bar{a}_{1p} \quad N/C$$

$P \rightarrow$ position of any other charge around Q_1

Electric Field Due to Discrete Charges



Consider n charges $Q_1, Q_2, Q_3, Q_4 \dots Q_N$. The field intensity is to be obtained at point 'P'. The distance of point 'P' from $Q_1, Q_2, Q_3, Q_4 \dots Q_N$ are $R_1, R_2, R_3, R_4 \dots R_n$. The unit vector along these directions are $\bar{a}_{R1}, \bar{a}_{R2}, \bar{a}_{R3}, \bar{a}_{R4} \dots \bar{a}_{Rn}$.

The total electric field intensity at point P is the vector sum of the individual field intensity produced by the various charges at P.

$$\bar{E} = \bar{E_1} + \bar{E_2} + \bar{E_3} + \bar{E_4} \cdots\cdots E_n$$

$$= \frac{Q_1}{4\pi\varepsilon_0 R_1^2}\bar{a}_{R1} + \frac{Q_2}{4\pi\varepsilon_0 R_2^2}\bar{a}_{R2} \cdots\cdots + \frac{Q_n}{4\pi\varepsilon_0 R_n^2}\bar{a}_{Rn}$$

$$\boxed{\bar{E} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{Q_i}{R_i^2}\bar{a}_{Ri}}$$

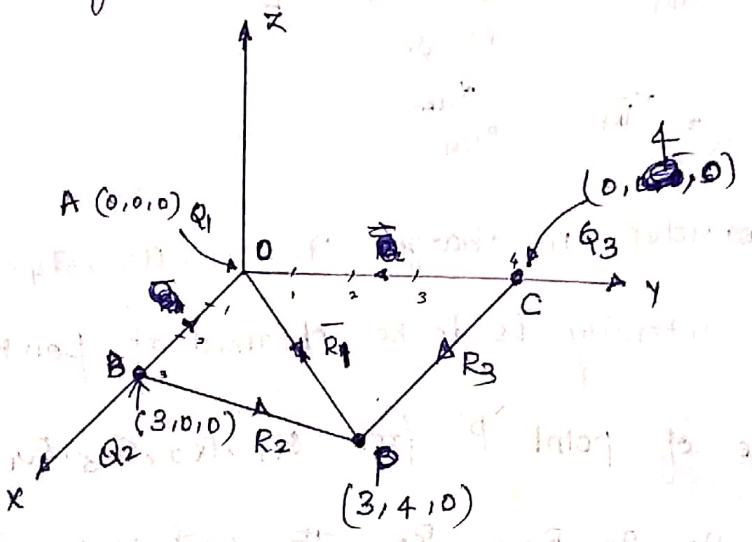
$$\bar{a}_{Ri} = \frac{\bar{r}_p - \bar{r}_1}{|\bar{r}_p - \bar{r}_i|}$$

$r_p \rightarrow$ position of vector point 'P'

$r_i \rightarrow$ position vector of point where Q_i charge is placed.

Problem (14)

Three point charges in free space are located as follows : 50 nC at (0,0) m, 40 nC at (3,0) m, − 60 nC at (0,4) m. Find the electric field intensity at (3,4) m.



$$E_p = \bar{E_1} + \bar{E_2} + \bar{E_3}$$

$$= \frac{Q_1}{4\pi\varepsilon_0 R_1^2}\bar{a}_{R1} + \frac{Q_2}{4\pi\varepsilon_0 R_2^2}\bar{a}_{R2} + \frac{Q_3}{4\pi\varepsilon_0 R_3^2}\bar{a}_{R3}$$

$\bar{R_1} = 3\bar{a}_x + 4\bar{a}_y$ $\quad \bar{R_2} = 4\bar{a}_y$

$\bar{R_3} = 3\bar{a}_x$ $\quad |\bar{R_1}| = 5$

$$\bar{E}_p = \frac{1}{4\pi\varepsilon_0}\left[\frac{50\times10^{-9}}{5^2}\left(\frac{3\bar{a}_x+4\bar{a}_y}{5}\right)+\frac{40\times10^{-9}}{4^2}\left(\frac{4\bar{a}_y}{4}\right)+\frac{-60\times10^{-9}}{3^2}\left(\frac{3\bar{a}_x}{3}\right)\right]$$

$$= \frac{10^{-9}}{4\pi\times8.854\times10^{-12}}\left[1.2\,\bar{a}_x+1.6\,\bar{a}_y+2.5\,\bar{a}_y-6.667\,\bar{a}_x\right]$$

$$\boxed{\bar{E}_p = -49.136\,\bar{a}_x + 36.85\,\bar{a}_y \quad V/m}$$

x ——— x

Electric Field Intensity due to various charges Distributions :-

(i) $\bar{E}$ due to line charge.

$$\bar{E} = \int_L \frac{\rho_L\, dl}{4\pi\varepsilon_0 R^2}\,\bar{a}_R$$

(ii) $\bar{E}$ due to Surface charge.

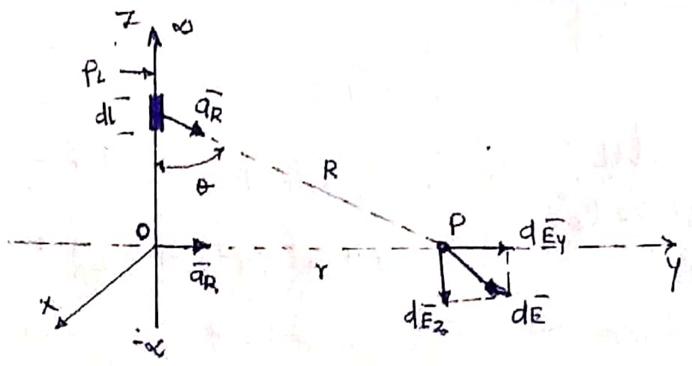
$$\bar{E} = \int_S \frac{\rho_s\, ds}{4\pi\varepsilon_0 R^2}\,\bar{a}_R$$

(iii) $\bar{E}$ due to Volume charge

$$\bar{E} = \int_V \frac{\rho_v \cdot dv}{4\pi\varepsilon_0 R^2}\,\bar{a}_R$$

x ——— x

Electric Field due to Infinite Line charge



Consider an infinitely long straight line.

carrying uniform line charge having density ρ_L c/m. Let this line lies along z axis from $-\infty$ to ∞ and hence called infinite line charge.

Let point 'P' is on y axis at which electric field intensity is to be determined.

consider a small differential length dl carrying a charge dQ, along z-axis, $dl = dz$

$$dQ = \rho_L \cdot dL = \rho_L . dz \quad \text{—— (1)}$$

Let co-ordinates of dQ $(0,0,z)$ and $P(0, r, 0)$

Then $\bar{R} = \bar{r_p} - \bar{r_{dl}}$

$$= [r\,\bar{a_y} - z\,\bar{a_z}]$$

$$|R| = \sqrt{r^2 + z^2}$$

$$\bar{a_R} = \frac{\bar{R}}{|\bar{R}|} = \frac{r\,\bar{a_y} - z\,\bar{a_z}}{\sqrt{r^2+z^2}} \quad \text{—— (2)}$$

$$d\bar{E} = \frac{dQ}{4\pi\varepsilon_0 R^2}\,\bar{a_R} = \frac{\rho_L\,dz}{4\pi\varepsilon_0\left(\sqrt{r^2+z^2}\right)}\left[\frac{r\,\bar{a_y} - z\,\bar{a_z}}{\sqrt{r^2+z^2}}\right] \quad \text{—— (3)}$$

For every charge on +ve x-axis there is equal charges present on -ve z-axis. Hence the z-component of electric field intensity produced by such charges at point P will cancel eachother. Hence the eqn $d\bar{E}$,

$$d\bar{E} = \frac{\rho_L \, dz}{4\pi \varepsilon_0 \left(\sqrt{r^2+z^2}\right)^2} \times \frac{r \bar{a_y}}{\sqrt{r^2+z^2}}$$

By integrating,

$$\bar{E} = \int_{-\infty}^{\infty} \frac{\rho_L}{4\pi\varepsilon_0 (r^2+z^2)^{3/2}} \, r . dz . \bar{a_y}$$

Where $z = r \tan\theta$, $r = \frac{z}{\tan\theta}$

$$dz = r \sec^2\theta \, d\theta$$

for $z = -\infty$, $\theta = \tan^{-1}(-\infty) = -\pi/2 = -90^\circ$

$z = \infty$, $\theta = \tan^{-1}(\infty) = \pi/2 = 90^\circ$

$$\therefore \bar{E} = \int_{\theta=-\pi/2}^{\pi/2} \frac{\rho_L \times r \times r \sec^2\theta \, d\theta}{4\pi\varepsilon_0 [r^2 + r^2\tan^2\theta]^{3/2}} . \bar{a_y}$$

$$\bar{E} = \frac{\rho_L}{4\pi\varepsilon_0} \int_{\theta=-\pi/2}^{\pi/2} \frac{r^2 \sec^2\theta \, d\theta}{r^3 [1+\tan^2\theta]^{3/2}} . \bar{a_y}$$

$1 + \tan^2\theta = \sec^2\theta$.

$$\bar{E} = \frac{\rho_L}{4\pi\varepsilon_0} \int_{\theta=-\pi/2}^{\pi/2} \frac{r^2 \sec^2\theta \, d\theta}{r^3 . \sec^3\theta} . \bar{a_y}$$

$$\bar{E} = \frac{\rho_L}{4\pi\varepsilon_0 r} \int_{\theta=-\pi/2}^{\pi/2} \frac{1}{\sec\theta} . \bar{a_y}$$

$$\frac{1}{\sec\theta} = \cos\theta$$

$$\therefore \bar{E} = \frac{\rho_L}{4\pi\epsilon_0 r} \int_{\theta=-\pi/2}^{\pi/2} \cos\theta \, d\theta \cdot \bar{a}_y$$

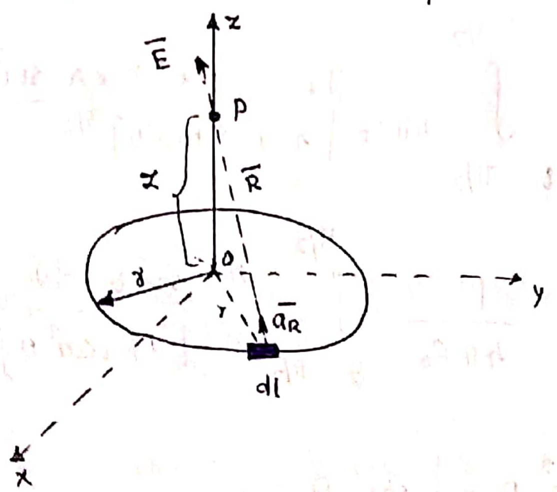
$$= \frac{\rho_L}{4\pi\epsilon_0 r} \left[-\sin\theta\right]_{\theta=-\pi/2}^{\pi/2} \cdot \bar{a}_y$$

$$= \frac{\rho_L \times \cancel{2}}{\cancel{4}_2 \pi \epsilon_0 r} \times \bar{a}_y$$

$$\boxed{E = \frac{\rho_L}{2\pi\epsilon_0 r} \cdot \bar{a}_y \ V/m}$$

Electric Field Due to charged Circular Ring

consider a charged circular ring of radius r placed in xy plane with centre at origin, carrying a charge uniformly along its circumferen. The charge density ρ_L C/m.



The point 'P' is at perpendicular distance 'z' from the ring. consider a small differential length dl on this ring. The charge on it is dQ

$$dQ = \rho_L \, dL$$

$$d\bar{E} = \frac{\rho_L \cdot dL}{4\pi \varepsilon_0 R^2} \bar{a_R}$$

$R \rightarrow$ distance b/w P from dl

$$dl = r \, d\varphi$$

$$R^2 = r^2 + z^2$$

$\bar{R}$ can be obtained from two components.

(i) Distance r in the direction of $-\bar{a_r}$, radically in wards, ie $-r\,\bar{a_r}$.

(ii) Distance z in the direction of $\bar{a_z}$, ie $z\,\bar{a_z}$

$$\therefore \bar{R} = -r\,\bar{a_r} + z\,\bar{a_z}$$

$$|\bar{R}| = \sqrt{(-r)^2 + z^2} = \sqrt{r^2 + z^2}$$

$$\bar{a_r} = \frac{\bar{R}}{|\bar{R}|} = \frac{-r\,\bar{a_r} + z\,\bar{a_z}}{\sqrt{r^2 + z^2}}$$

$$\therefore d\bar{E} = \frac{\rho_L \, dl}{4\pi \varepsilon_0 \left(\sqrt{r^2+z^2}\right)^2} \times \frac{-r\,\bar{a_r} + z\,\bar{a_z}}{\sqrt{r^2+z^2}}$$

$$d\bar{E} = \frac{\rho_L \, (r\,d\varphi)}{4\pi \varepsilon_0 \, (r^2+z^2)^{3/2}} \times \left[-r\,\bar{a_r} + z\,\bar{a_z}\right]$$

The radial components of $\bar{E}$ at point p will be symmetrically placed in the plane parallel to xy plane and are going to cancel each other.

$$\therefore d\bar{E} = \frac{\rho_L \, (r\,d\varphi)}{4\pi \varepsilon_0 \, (r^2+z^2)^{3/2}} \cdot z\,\bar{a_z}$$

$$\bar{E} = \int_{\varphi=0}^{2\pi} \frac{\rho_L \, r \, d\varphi}{4\pi \varepsilon_0 \, (r^2+z^2)^{3/2}} \times z \cdot \bar{a_z}$$

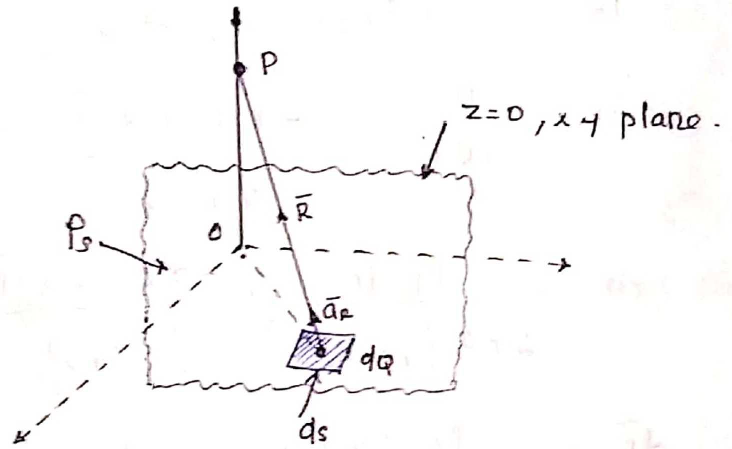
$$= \frac{\rho_L\, r\, z \quad \bar{a}_z}{4\pi\varepsilon_0 (r^2+z^2)^{3/2}} \cdot \left[\phi\right]_0^{2\pi}$$

$$\boxed{\bar{E} = \frac{\rho_L\, r.\, z}{2\,\varepsilon_0\,(r^2+z^2)^{3/2}}\,\bar{a}_z}$$ //

Electric Field Due to Infinite sheet of Charge

Consider an infinite sheet of charge having uniform charge density ρ_s C/m^2. placed in xy plane. Let us use cylinderical co ordinates.

The point P at which $\bar{E}$ to be calculated is on z-axis.



Consider the differential surface area ds carrying a charge dQ. The normal direction to ds is z-direction hence ds normal to z direction is $r\,dr\,d\phi$.

$$dQ = \rho_s\, ds = \rho_s\, r\, dr\, d\phi$$

$$d\bar{E} = \frac{dQ}{4\pi\varepsilon_0 R^2}\,\bar{a}_R = \frac{\rho_s\, r\, dr\, d\phi}{4\pi\varepsilon_0 R^2}\,\bar{a}_R$$

The distance vector $\bar{R}$ has two components as

(i) The radial component r along $-\bar{a}_r$, ie $-r\bar{a}_r$

2) The component x along a_x axis is $z\,\bar{a}_z$

$\therefore$ $$\bar{R} = -r\,\bar{a}_r + z\,\bar{a}_z$$

$$|\bar{R}| = \sqrt{r^2+z^2}$$

$$\bar{a}_R = \frac{\bar{R}}{|\bar{R}|} = \frac{-r\,\bar{a}_r + z\,\bar{a}_z}{\sqrt{r^2+z^2}}$$

$$d\bar{E} = \frac{\rho_s\, r\, dr\, d\varphi}{4\pi\varepsilon_0\left(\sqrt{r^2+z^2}\right)^2}\left[\frac{\left(-r\bar{a}_r + z\,\bar{a}_z\right)}{\sqrt{r^2+z^2}}\right]$$

r varies from 0 to ∞

φ varies from 0 to 2π

$$\bar{E} = \int_{\phi=0}^{2\pi}\int_{r=0}^{\infty} d\bar{E} = \int_0^{2\pi}\int_0^{\infty}\frac{\rho_s\, r\, dr\, d\varphi}{4\pi\varepsilon_0\,(r^2+z^2)^{3/2}}\cdot z\,\bar{a}_z$$

put $r^2+z^2=u^2$, $2r\,dr = 2u\,du$

$r=0,\ u=z,\quad r=\infty,\ u=\infty.$

$$\bar{E} = \int_0^{2\pi}\int_{u=z}^{\infty}\frac{\rho_s}{4\pi\varepsilon_0}\ \frac{u\,du}{4\pi\varepsilon_0\,(u^2)^{3/2}}\ d\varphi\ z.\bar{a}_z$$

$$= \int_0^{2\pi}\int_{u=z}^{\infty}\frac{\rho_s}{4\pi\varepsilon_0}\ \frac{du}{u^2}\ d\varphi\ z.\bar{a}_z$$

$$= \int_0^{2\pi}\frac{\rho_s}{4\pi\varepsilon_0}\,d\varphi\ z.\bar{a}_z\left[-\frac{1}{u}\right]_z^{\infty}$$

$$\int\frac{1}{u^2} = \int u^{-2} = \frac{u^{-1}}{-1} = -\frac{1}{u}$$

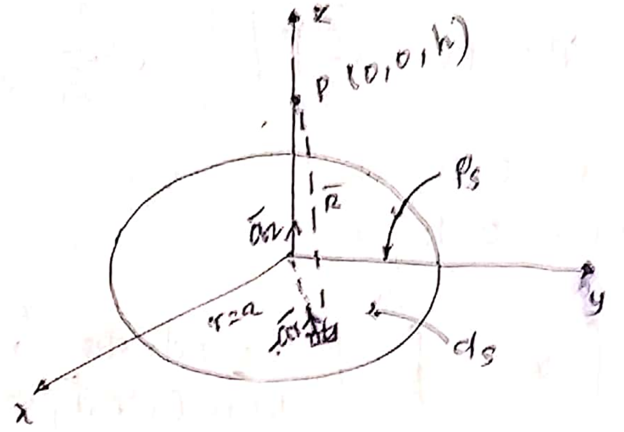
$$= \frac{\rho_s}{4\pi\varepsilon_0}\left[\varphi\right]_0^{2\pi}\ z.\bar{a}_z\left[-\frac{1}{\infty} - \left(-\frac{1}{z}\right)\right]$$

$$= \frac{\rho_s}{4\pi\varepsilon_0}\,2\pi.\bar{a}_z$$

$$\boxed{\bar{E} = \frac{\rho_s}{2\pi\varepsilon_0}\,\bar{a}_z \quad V/m}$$

—— x —— x ——

Problem (15) Find the force on a point charge q located at $(0,0,h)$ m due to charge of surface charge density ρ_s C/m^2 uniformly distributed over the circular disc $r \le a$, $z=0$ m. Also find the electric field intensity at the same point.



Consider the differential area ds carrying the charge dQ. The normal direction to ds is $\vec{a_z}$, hence

$$ds_z = r\,dr\,d\phi$$

$$dQ = \rho_s\,ds$$

$$= \rho_s \times r . dr . d\phi$$

The force on a point charge q due to dq is

$$\overline{dF} = \frac{q . dQ}{4\pi\varepsilon_0 R^2}\,\overline{a_R}$$

$\overline{R}$ can be splited

1) The component along $-\bar{a}_r$ having length r, ie $-r\,\overline{a_r}$

2) The component $z=h$ along $\overline{a_z}$

$$\therefore \hat{R} = -r\,\bar{a_r} + h\,\bar{a}_z$$

$$|\bar{R}| = \sqrt{r^2+h^2}$$

$$\bar{a}_r = \frac{\bar{R}}{|\bar{R}|} = \frac{-r\,\bar{a}_r + h\,\bar{a}_z}{\sqrt{r^2+h^2}}$$

$$d\bar{F} = \frac{dQ\,q}{4\pi\varepsilon_0\sqrt{r^2+h^2}} \quad \frac{-r\,\bar{a}_r + h\,\bar{a}_z}{\sqrt{r^2+h^2}}$$

Due to symmetry about z-axis, all radial components will cancel each other.

$$\bar{F} = \int d\bar{F} = \int_{\varphi=0}^{2\pi}\int_{r=0}^{a} \frac{\rho_s\, r\,dr\,d\varphi\, q}{4\pi\varepsilon_0\,(r^2+h^2)^{3/2}}\, .\, h\,\bar{a}_z$$

$$r^2+h^2 = u^2, \quad 2r\,dr = 2u\,du$$

$$r=0,\ u_1 = h, \quad r=a,\ u_2 = \sqrt{a^2+h^2}$$

$$\bar{F} = \int_{\varphi=0}^{2\pi}\int_{u_1}^{u_2} \frac{\rho_s q}{4\pi\varepsilon_0}\, \frac{u\,.du}{(u^2)^{3/2}}\, d\varphi\, h\,\bar{a}_z$$

$$= \frac{\rho_s h q}{4\pi\varepsilon_0}\int_{\varphi=0}^{2\pi}\int_{u_1}^{u_2} \frac{1}{u^2}\, du\, d\varphi\, \bar{a}_z$$

$$= \frac{\rho_s h q}{4\pi\varepsilon_0}\,[\phi]_0^{2\pi}\left[-\frac{1}{u}\right]_{u_1}^{u_2} . \bar{a}_z$$

$$= \frac{q\rho_s h}{4\pi\varepsilon_0}(2\pi)\left[\frac{-1}{u_2} + \frac{1}{u_1}\right]$$

$$\bar{F} = \frac{q\rho_s h}{2\varepsilon_0}\left[-\frac{1}{\sqrt{a^2+h^2}} + \frac{1}{h}\right]\bar{a}_z \ . \ N$$

The electric field intensity

$$E = \bar{F}/\text{charge}.$$

$$\boxed{\bar{E} = \frac{\rho_s h}{2\varepsilon_0}\left[\frac{1}{h} - \frac{1}{\sqrt{a^2+h^2}}\right] . \bar{a}_z \ V/m}$$

Electric Flux Density (D)

The net flux passing normal through the unit surface area is called the electric flux density.

$$D = \frac{\psi}{S}$$

$\psi \rightarrow$ total flux , $S \rightarrow$ total surface area

$\bar{D}$ due to a point charge Q

In the vector form, electric flux density at a point which is at a distance of r, from the point charge +Q is given by

$$\bar{D} = \frac{Q}{4\pi r^2} \bar{a_r} \quad C/m^2$$

Relationship between $\bar{D}$ and $\bar{E}$

W.K.T

$$\bar{E} = \frac{Q}{4\pi \epsilon_0 r^2} \bar{a_r} \quad \& \quad \bar{D} = \frac{Q}{4\pi r^2} \bar{a_r}$$

$$\frac{\bar{D}}{\bar{E}} = \frac{\frac{Q}{4\pi r^2} \bar{a_r}}{\frac{Q}{4\pi \epsilon_r r^2} , \bar{a_r}} = \epsilon_0 .$$

$$\boxed{\bar{D} = \epsilon_0 \bar{E}} \text{, for free space.}$$

or $\bar{D} = \epsilon_0 \epsilon_r \bar{E}$

Electric Flux Density for various charge Distribution

(i) Line charge

$$\bar{D} = \frac{\rho_L}{2\pi r} \bar{a_r}$$

(ii) Surface charge:

$$\vec{D} = \frac{\rho_s}{2}\,\vec{a}_n$$

(iii) Volume charge:

$$\vec{D} = \frac{\int_v \rho_v\,dv}{4\pi r^2}\,\vec{a}_r$$

Gauss's Law

The electric flux passing through any closed surface is equal to the total charge enclosed by that surface.

$$\Psi = Q = \oint_S \vec{D}\cdot d\vec{s} = \int_v \rho_v\,dv$$

Limitations:

- It is applicable to symmetrical problems only.
- It is applicable only on Gaussian surface.
- It can be applied only if the surface encloses the volume completely.

Applications of Gauss's law

(ii) Surface charge :-

$$\bar{D} = \frac{\rho_s}{2} \bar{a_n}$$

(iii) Volume charge :-

$$\bar{D} = \frac{\int_v \rho_v \, dv}{4\pi r^2} \bar{a_r}$$

Gauss's Law

The electric flux passing through any closed surface is equal to the total charge enclosed by that surface.

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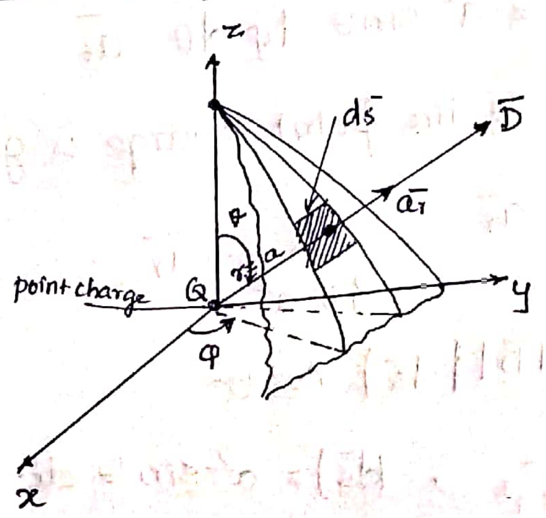
Applications of Gauss's law

(i) Point charge



Let a point charge Q is located at the origin.

To determine $\bar{D}$ and to apply Gauss's law..

Consider a spherical surface around Q with centre as origin.

The spherical surface is gaussian surface and it satisfied required condition.

* $\bar{D}$ is always directed radially outwards along $\bar{a_r}$ which is normal to the spherical surface at any point 'P' on the surface.

Consider a differential surface area ds. and its direction is $\bar{a_r}$, considering spherical co-ordinate system.

radius of the sphere $r = a$.

Spherical co ordinate system

$$ds = r^2 \sin\theta \, d\theta \, d\varphi$$

$$= a^2 \sin\theta \, d\theta \, d\varphi$$

$$\bar{ds} = ds . \bar{a_n} = a^2 \sin\theta \, d\varphi \, d\theta , \bar{a_r}$$

Now $\bar{D}$ due to the point charge is given by

$$\bar{D} = \frac{Q}{4\pi r^2} \bar{a_r} = \frac{Q}{4\pi a^2} \bar{a_r}$$

$$\bar{D} \cdot \bar{ds} = |\bar{D}| |\bar{ds}| \cos\theta$$

$$|\bar{D}| = \frac{Q}{4\pi a^2} , \quad |\bar{ds}| = a^2 \sin\theta \, d\theta \, d\varphi$$

$$\theta = 0$$

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$\bar{D}$ & $d\bar{s}$ are along the $\bar{a_r}$. so angle b/w both $\bar{D}$ & $d\bar{s}$ is zero.

$$\therefore \ \bar{D} \cdot d\bar{s} = \frac{Q}{4\pi a^2} \times a^2 \sin\theta \, d\theta \, d\varphi \, \cos 0^\circ$$

$$\bar{D} \cdot d\bar{s} = \frac{Q}{4\pi} \sin\theta \, d\theta \, d\varphi$$

Alternatively,

$$\bar{D} \cdot d\bar{s} = \frac{Q}{4\pi a^2} \bar{a_r} \cdot a^2 \sin\theta \, d\theta \, d\varphi \, \bar{a_r}$$

$$= \frac{Q}{4\pi} \sin\theta \, d\theta \, d\varphi \, [\underbrace{\bar{a_r} \cdot \bar{a_r}}_{1}]$$

$$\therefore \ \bar{D} \cdot d\bar{s} = \frac{Q}{4\pi} \sin\theta \, d\theta \, d\varphi$$

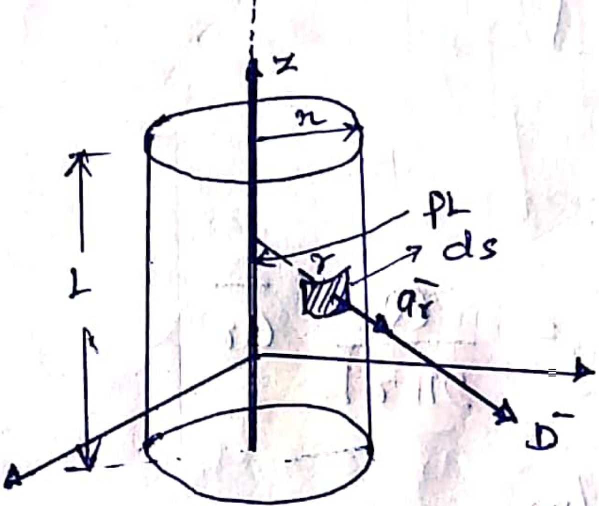
$$\psi = \oint_S \bar{D} \cdot d\bar{s} = \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} \frac{Q}{4\pi} \sin\theta \, d\theta \, d\varphi$$

$$= \frac{Q}{4\pi} [-\cos\theta]_0^{\pi} \cdot [\varphi]_0^{2\pi}$$

$$= \frac{Q}{4\pi} [-(-1) - (-1)] [2\pi]$$

$$\boxed{\psi = Q}$$ $\rightarrow$ Gauss's law proved.

Infinite Line Charges:-



The line charge along z-axis, $\bar{D}$ not in z-axis direction. $\bar{D}$ has only radial component.

Now $Q = \oint_s \bar{D} \cdot d\bar{s}$

$$Q = \oint_{side} \bar{D} \cdot d\bar{s} + \oint_{top} \bar{D} \cdot d\bar{s} + \oint_{bottom} \bar{D} \cdot d\bar{s}$$

$\bar{D} = \bar{D_r} \cdot \bar{a_r}$ (radial component only)

$d\bar{s} = r\, d\varphi\, dz\, \bar{a_r}$ $\rightarrow \bar{a_r}$ direction

$\bar{D} \cdot d\bar{s} = D_r \cdot r\, d\varphi\, dz\, (\bar{a_r} \cdot \bar{a_r})$

$$\boxed{\bar{D} \cdot d\bar{s} = D_r \cdot r\, d\varphi\, dz}$$

$D_r \rightarrow$ constant over the side surface.

$\bar{D}$ has only radial component.

Hence integrations over top and bottom surfaces is zero.

$$\oint_{top} \bar{D} \cdot d\bar{s} = \oint_{bottom} \bar{D} \cdot d\bar{s} = 0$$

$$Q = \oint_{side} \bar{D} \cdot d\bar{s} = \oint_{side} D_r \cdot r \cdot d\varphi\, dz$$

$$= \int_{z=0}^{L} \int_{\varphi=0}^{2\pi} D_r \cdot r \cdot d\varphi\, dz$$

$$= r \cdot D_r \cdot [z]_0^L \cdot [\varphi]_0^{2\pi}$$

$$\boxed{Q = 2\pi r \cdot D_r \cdot L}$$

$$D_r = \frac{Q}{2\pi r \cdot L}$$

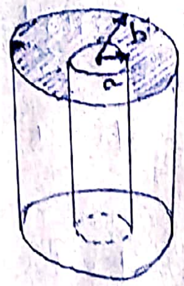
$$\bar{D} = D_r \cdot \bar{a_r} = \frac{Q}{2\pi r L} \bar{a_r}$$

$$\frac{Q}{L} = \rho_L \quad C/m$$

$$\boxed{\bar{D} = \frac{\rho_L}{2\pi r} \bar{a_r} \quad C/m^2}$$

$$\bar{E} = \frac{\bar{D}}{\varepsilon_0} = \frac{\rho_L}{2\pi\varepsilon_0} \bar{a_r} \quad V/m.$$

Co-axial Cable:



The charge distribution on the outer surface of the inner conductor is having density ρ_s c/m². The total outer surface area of the inner conductor is $2\pi a L$. Hence ρ_s can be expressed interms of ρ_L.

$$\rho_L = \frac{\rho_s \times \text{surface area}}{\text{Total length}} = \frac{\rho_s \times 2\pi a L}{L}$$

$$\boxed{\rho_L = 2\pi a \rho_s \quad C/m}$$

$\bar{D}$ has only radial component, then line charge

$$Q = D_r \cdot 2\pi r L \quad \text{——— (1)}$$

where $a < r < b$

Total charge in inner conductor,

$$Q = \oint_S \rho_s \, d\bar{s}$$

$$ds = r \, d\phi \, dz \quad \text{but } r = a$$

$$ds = a \, d\phi \, dz$$

$$Q = \oint_S \rho_s \, a \, d\phi = \int_{z=0}^{L} \int_{\phi=0}^{2\pi} \rho_s . a . d\phi . dz$$

$$= \rho_s . a . [z]_0^L . [\phi]_0^{2\pi}$$

$$\boxed{Q = 2\pi a L \rho_s} \quad \text{——— (2)}$$

equate (1) & (2)

$$D_r \cdot 2\pi r L = 2\pi a L \rho_s$$

$$D_r = \frac{a \rho_s}{r}$$

This act along radial direction

$$\bar{D} = \frac{a \rho_s}{r} \bar{a_r}$$

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But $\rho_s = \dfrac{\rho_L}{2\pi a}$

$\therefore \quad \bar{D} = \dfrac{a\,\rho_L}{2\pi a\, r}\,\bar{a_r}$

$$\boxed{\bar{D} = \frac{\rho_L}{2\pi r}\,\bar{a_r}}$$

$$\boxed{\bar{E} = \frac{\rho_L}{2\pi \epsilon_0 r}\,\bar{a_r}, \quad a < r < b \quad V/m}$$

and total charge at

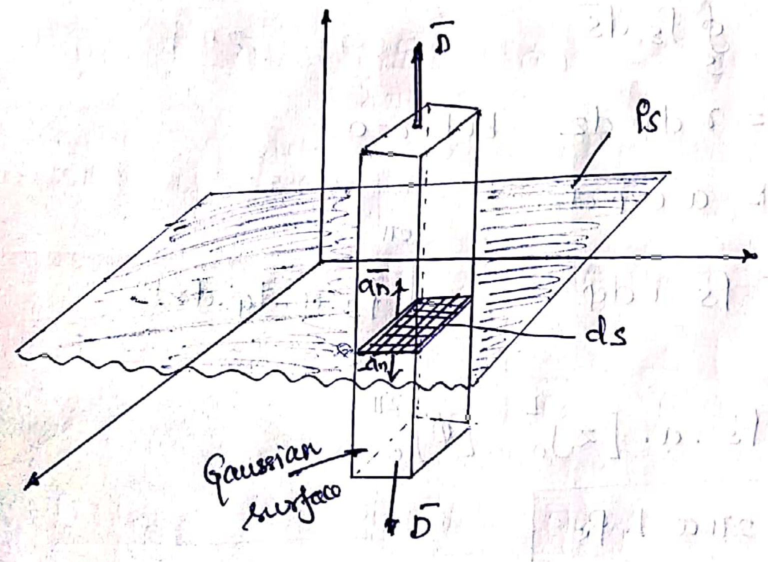
$Q_{\text{outer cylinder}} = -2\pi a L \rho_s \text{ (inner)}$

$Q_{\text{outer cylinder}} = 2\pi b L \rho_s \text{ (outer)}$

$-2\pi a L \rho_s = 2\pi b L \rho_s$

$\rho_s\text{(outer)} = -\dfrac{a}{b}\,\rho_s\text{ (inner)}.$

Infinite sheet of charge:-



$ds = dx \cdot dy$

$\bar{a_n} = \bar{a_z}, \quad -\bar{a_n} = -\bar{a_z}$

$\bar{D} = 0$ in (x & y direction).

Hence charge enclosed

$$Q = \oint_S \bar{D}\cdot d\bar{s} = \oint_{\text{side}} \bar{D}\cdot d\bar{s} + \oint_{\text{top}} \bar{D}\cdot d\bar{s} + \oint_{\text{bottom}} \bar{D}\cdot d\bar{s}$$

$\oint_{\text{Side}} \bar{D}.d\bar{s} = 0$ $\quad$ $\bar{D}$ has no componone in x & y directione.

$\bar{D} = D_z \bar{a_z}$ for top surface.

$d\bar{s} = dx\,dy\,\bar{a_z}$

$\bar{D}\cdot d\bar{s} = D_z\,dx\,dy\,(\bar{a_z}\cdot\bar{a_z})$

$= D_z\,dx\,dy$

$\bar{D} = D_z(-\bar{a_z})$ bottom surface

$d\bar{s} = dx\,dy\,(-\bar{a_z})$

$\bar{D}\cdot d\bar{s} = D_z\,dx\,dy\,(\bar{a_z}\cdot\bar{a_z})$

$\bar{D}\cdot d\bar{s} = D_z\,dx\,dy$

$$Q = \oint_{top} D_z\,dx\,dy + \oint_{bottom} D_z\,dx\,dy$$

Now $\oint_{top} dx\,dy = \oint_{bottom} dx\,dy = A$ = Area of surface.

$\therefore\ Q = 2D_z A$

$Q = \rho_s A$

$\rho_s = 2D_z$

$D_z = \rho_s/2$

$$\boxed{\begin{aligned} \therefore\ \bar{D} &= D_z\bar{a_z} = \frac{\rho_s}{2}\bar{a_z}\ \ C/m^2 \\ \bar{E} &= \frac{\bar{D}}{\varepsilon_0} = \frac{\rho_s}{2\varepsilon_0}\bar{a_z}\ \ V/m. \end{aligned}}$$

Unit - II

Electro-Statics - II

Potential Difference :-

The workdone in moving a charge Q from point B to A in the electric field $\bar{E}$ is given by

$$W = -Q \int_B^A \bar{E} \cdot d\bar{l}$$

and

Potential difference $V = -\int_B^A \bar{E} \cdot \bar{dl}$ Volt.

$$1 \text{ Volt} = \frac{1 \text{ Joule}}{1 \text{ coulomb}}$$

Potential due to point charge :-

$$\bar{E} = \frac{Q}{4\pi\varepsilon_0 r^2} \hat{a_r}$$

For spherical system,

$$d\bar{l} \rightarrow dr\, \bar{a_r} + r\, d\theta\, \bar{a_\varphi} + r \sin\theta\, d\varphi\, \bar{a_\varphi}$$

Then,
$$V = \frac{Q}{4\pi\varepsilon_0} \left[\frac{1}{r_A} - \frac{1}{r_B} \right] \quad V$$

Potential Due to line Charges

$$V_A = \int_l \frac{\rho_L(r')\, dl'}{4\pi\varepsilon_0 R} \quad V$$

Potential due to surface Charge :-

$$V_A = \int_S \frac{\rho_s(r')\, ds'}{4\pi\varepsilon_0 R} \quad V$$

Potential due to Volume Charge

$$V_A = \int_V \frac{\rho_v (r') \, dv'}{4\pi \epsilon_0 R} \quad V$$

---

problem ①

Find the electric potential at any point given the electric field $\bar{E} = \frac{2r}{(r^2+a^2)^2} \, \bar{a_r}$

The boundary conditions are : at $r = \infty$, $V = 0$ and at $r = 0$ and $V = 100$

Solution:

The potential is given by,

$$V = -\int \bar{E} \cdot dl$$

$$\bar{E} = \frac{2r}{(r^2+a^2)^2} \cdot \bar{a_r}$$

Let $r^2+a^2 = u$, $2r \cdot dr = du$.

then $\bar{E} = -\int \frac{du}{u^2} = -\int u^{-2} du$.

$$V = \frac{1}{u} + c$$

$$V = \frac{1}{r^2+a^2} + c$$

At $r = \infty$, $V = 0$, $0 = \frac{1}{\infty} + c$ ie $c = 0$

At $r = 0$, $V = 100$, $100 = \frac{1}{a^2} + c$ ie $c = 100 - \frac{1}{a^2}$

Hence the potential function is given by

$$V = \frac{1}{r^2+a^2} + 100 - \frac{1}{a^2}$$

Potential Difference due to infinite line charge

$$V_{AB} = \frac{\rho_L}{2\pi \epsilon_0} \ln\left(\frac{r_B}{r_A}\right) \quad V$$

Equipotential Surface

An equippotential surface is an imaginary surface in an electric field of a given charge distribution, in which all the points on the surface are at the same electric potential.

The potential difference at any two points on the equipotential surface is always zero.

For uniform field $\bar{E}$, the equipotential surfaces are perpendicular to $\bar{E}$ and are equispaced for fixed increment of voltages. Thus $\bar{E}$ and equipontential surface are at right angles to each other.

For non-uniform field, the field lines tends to diverge in the direction of decreasing $\bar{E}$. Hence $\bar{E}(\bar{a_r}) \perp r\ \bar{E}$ but not equispaced, for fixed increment of voltages.

Conservative Field :-

$$\oint_{\text{Closed path}} \bar{E} \cdot d\bar{l} = 0 .$$

Potential Gradient

$$\frac{dV}{dL} = \lim_{\Delta L \to 0} \frac{\Delta V}{\Delta L} = \text{potential gradient.}$$

Relation between $\bar{E}$ and V

$$\bar{E} = -\left.\frac{dV}{dL}\right|_{max} \bar{a}_n$$

$$\bar{E} = -\nabla V = -(\text{grad } V)$$

Vector operator

S.No	Coordinate system	Grad V = ∇V
1.	cartesian	$\nabla V = \frac{\partial V}{\partial x} \bar{a}_x + \frac{\partial V}{\partial y} . \bar{a}_y + \frac{\partial V}{\partial z} . \bar{a}_z$
2.	cylinderical	$\nabla V = \frac{\partial V}{\partial r} . \bar{a}_r + \frac{1}{r} \frac{\partial V}{\partial \phi} \bar{a}_\phi + \frac{\partial V}{\partial z} \bar{a}_z$
3.	spherical	$\nabla V = \frac{\partial V}{\partial r} \bar{a}_r + \frac{1}{r} \frac{\partial V}{\partial \theta} \bar{a}_\theta + \frac{1}{r\sin\theta} \frac{\partial V}{\partial \phi} \bar{a}_\phi$

Energy Density in the Electrostatic Fields.

$$W_E = \frac{1}{2} \sum_{m=1}^{n} Q_m V_m \quad J$$

Energy stored interms of $\bar{D}$ and $\bar{E}$

$$W_E = \frac{1}{2} \int_V \bar{D} . \bar{E} \, dv$$

Energy density,

$$\frac{dW_E}{dv} = \frac{1}{2} \bar{D} . \bar{E} \quad J/m^3$$

$$W_E = \int_V \frac{dW_E}{dv} . dv .$$

An Electric Dipole

Two point charges of equal magnitude but opposite sign, seperated by a very small distance give rise to an electric dipole.

Current

The flow of charge per unit time.

$$I = \frac{dQ}{dt} \quad C/s.$$

Current density

The current passing through the unit surface area, when the surface is held normal to the direction of current.

Relation between I and J

$$I = \int_S J \cdot ds.$$

Relation between J and ρ_v

$$\bar{J} = \rho_v \cdot \bar{v} \qquad \bar{v} \rightarrow \text{velocity vector.}$$

Continuity Equation:

$$\nabla \cdot \bar{J} = -\frac{\partial \rho_v}{\partial t}$$

Steady current

$$\nabla \cdot \bar{J} = 0 \quad \text{(Steady current)}$$

Conductor:

$$\bar{J} = \sigma \bar{E}$$

$\sigma \rightarrow$ conductivity.

Resistivity is the reciprocal of the conductivity.

So as temp. increases, the conductivity decreases and resistivity increases.

Resistance of a conductor

$$R = \frac{V}{I} = \frac{L}{\sigma S}$$

Properties of conductor :-

* Under the static conditions, no charge and no electric field can exist at any point with in the conducting material.

* The charge can exist on the surface of the conductor giving rise to surface charge density.

* Within a conductor, the charge density is always zero.

* The charge distribution on the surface depends on the shape of the surface.

* The conductivity of an ideal conductor is infinite.

* The conductor surface is an equipotential surface.

Dielectric Materials:-

* donot have free charges.

* charges are bounded by the infinite force

Electric dipoles produce an electric field which opposes the externally applied electric field. Due to which seperation of bound charges results to produce electric dipoles, under influence of electric field $\vec{E}$, is called polarization.

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Dielectric Strength:

The minimum value of applied electric field at which the dielectric breaks down is called dielectric strength of that dielectric.

Utilization Factor:

Ratio of the avg. electric field to the max. value of an electric field.

$$\eta = \text{Utilization factor} = \frac{E_{avg}}{E_{max}}$$

The reciprocal of utilization factor is called inhomogenity of an electric field.

Boundary Conditions

The conditions existing at the boundary of the two media when field passes from one medium to another medium are called boundary conditions.

Depending upon the nature of the media, there are two situations;

* Boundary between conductor and free space
* Boundary between two dielectrics with different properties.

$$\oint \bar{E} \cdot \bar{dl} = 0 \qquad \oint \bar{D} \cdot \bar{ds} = Q$$

$$\bar{E} = \bar{E}_{tan} + \bar{E}_N$$

Boundry conditions between conductor and free space

1) The field intensity inside a conductor is zero and the flux density inside a conductor is zero.

2) No charge can exist within a conductor. The charge appears on the surface in the form of surface charge density.

3) The charge density within the conductor is zero.

$\bar{E}$ at the boundry.

1) The component tangential to the surface ($\bar{E}_{tan}$)

2) The component normal to the surface ($\bar{E}_N$).

$$E_{tan} = 0$$

D_N at the boundry:-

$$D_N = \rho_s$$

$$E_N = \frac{\rho_s}{\varepsilon_0}.$$

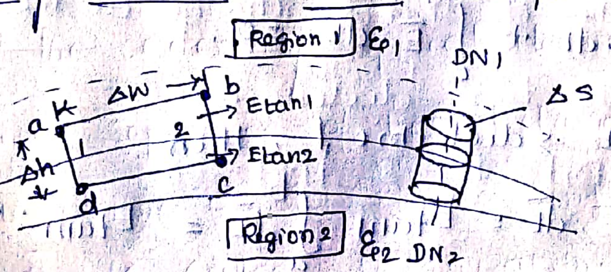
Boundry conditions between conductor and Dielectric

$$E_{tan} = D_{tan} = 0$$

$$D_N = \rho_s$$

$$E_N = \frac{\rho_s}{\varepsilon} = \frac{\rho_s}{\varepsilon_0 \varepsilon_r}$$

Boundry condition Between two perfect Dielectrics



Consider closed path abcda rectangular in shape having elementry heigh Δh, elementry width ΔW. It is placed at

Dielectric 1 $= \Delta h/2$

Dielectric 2 $= \Delta h/2$

Let us evaluate the integral of $\bar{E}\cdot\bar{dL}$ along this path, tracing it in clockwise direction as a-b-c-d-a.

$$\oint \bar{E}\cdot\bar{dL} = 0.$$

$$\int_a^b \bar{E}\cdot\bar{dL} + \int_b^c \bar{E}\cdot\bar{dL} + \int_c^d \bar{E}\cdot\bar{dL} + \int_d^a \bar{E}\cdot\bar{dL} = 0 \quad \text{--- (1)}$$

Now $\bar{E_1} = \bar{E}_{1t} + \bar{E}_{1N}$

$\bar{E_2} = \bar{E}_{2t} + \bar{E}_{2N}$

Both $\bar{E_1}$ and $\bar{E_2}$ in the respective dielectrics have both the components, normal & tangential

Let $|\bar{E}_{1t}| = E_{tan1}$, $|\bar{E}_{2t}| = E_{tan2}$

$|\bar{E}_{1N}| = E_{1N}$, $|\bar{E}_{2N}| = E_{2N}$

Now for rectangle to be reduced at the surface to analyse the boundary condition, $\Delta h \to 0$. then $\int_b^c$ & $\int_d^a$ become zero. Hence.

$$\int_a^b \bar{E}\cdot\bar{dL} + \int_c^d \bar{E}\cdot\bar{dL} = 0. \longrightarrow (2)$$

Now a-b → dielectric 1, then

$$\int_a^b \bar{E}\cdot\bar{dL} = E_{tan1}\int_a^b \bar{dL} = E_{tan1}\cdot\Delta W$$

Similarly, c-d → dielectric-2, but opposite to a-b

then,

$$\int_c^d \bar{E} \cdot \bar{dL} = -E_{tan2} \cdot \Delta W$$

sub. in eqn ②,

$$E_{tan1} \cdot \Delta W - E_{tan2} \cdot \Delta W = 0$$

$$\boxed{E_{tan1} = E_{tan2}}$$

The tangential components of field boundary in both dielectric remain same

and w.k.t, $\bar{D} = \epsilon \bar{E}$

$$D_{tan1} = \epsilon_1 E_{tan1}, \quad D_{tan2} = \epsilon_2 E_{tan2}$$

$$\frac{D_{tan1}}{\epsilon_1} = \frac{D_{tan2}}{\epsilon_2}$$

$$\boxed{\frac{D_{tan1}}{D_{tan2}} = \frac{\epsilon_1}{\epsilon_2}}$$

To find normal components using Gauss's Law, consider a Gaussian surface in the form of right circular cylinder, placed in such a way that half of it lies in dielectric 1 remaining in dielectric 2.

The heigh $\Delta h \to 0$, hence flux leaving from its lateral surface is zero. The surface area of its top & bottom is ΔS

$$\oint \bar{D} \cdot \bar{ds} = Q$$

$$\left[\int_{top} + \int_{bottom} + \int_{lateral}\right] \bar{D} \cdot \bar{ds} = Q.$$

But $\int_{Lateral} \bar{D} \cdot \bar{ds} = 0$ as $\Delta h \to 0$,

$$\therefore \int_{top} \bar{D} \cdot \bar{ds} + \int_{bottom} \bar{D} \cdot \bar{ds} = Q.$$

then $|\bar{D}| = D_{N_1}$

$$\int_{top} \bar{D} \cdot \bar{ds} = D_{N_1} \int_{top} \bar{ds} = D_{N_1} \cdot \Delta S.$$

$$\int_{bottom} \bar{D} \cdot \bar{ds} = -D_{N_2} \int_{bottom} \bar{ds} = -D_{N_2} \cdot \Delta S.$$

Then

$$D_{N_1} \cdot \Delta S - D_{N_2} \cdot \Delta S = Q.$$

But $Q = \rho_s \cdot \Delta s$

$$\therefore \boxed{D_{N_1} - D_{N_2} = \rho_s}$$

There is no free charge in perfect dielectric.

$\therefore$ charge density $\rho_s = 0$

$$\therefore D_{N_1} - D_{N_2} = 0$$

$$\boxed{D_{N_1} = D_{N_2}}$$

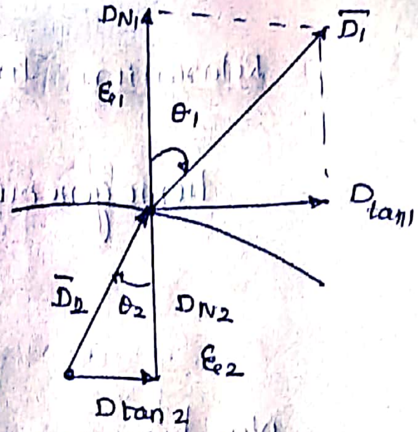
Now $D_{N_1} = \epsilon_1 E_{N_1}$, $D_{N_2} = \epsilon_2 E_{N_2}$

$$\therefore \boxed{\frac{E_{N_1}}{E_{N_2}} = \frac{\epsilon_2}{\epsilon_1}}$$

Refraction of $\bar{D}$ at the boundary

The direction of $\bar{D}$ & $\bar{E}$ change at the boundary between the two dielectrics.

Let $|\bar{D}_1| = D_1$ & $|\bar{D}_2| = D_2$



$D_{N1} = D_1 \cos\theta_1$

$D_{N2} = D_2 \cos\theta_2$

But $D_{N1} = D_{N2}$

$D_1 \cos\theta_1 = D_2 \cos\theta_2$

$$\frac{D_{tan1}}{D_{tan2}} = \frac{\epsilon_1}{\epsilon_2}$$

$D_{tan1} = D_1 \sin\theta_1$; $D_{tan2} = D_2 \sin\theta_2$

then $$\frac{D_1 \sin\theta_1}{D_2 \sin\theta_2} = \frac{\epsilon_1}{\epsilon_2} = \frac{D_{tan1}}{D_{tan2}}$$

$$\tan\theta_1 = \frac{D_{tan1}}{D_{N1}} \qquad \tan\theta_2 = \frac{D_{tan2}}{D_{N2}}$$

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{D_{tan1}}{D_{tan2}} \cdot \frac{D_{N2}}{D_{N1}}$$

$$\boxed{\frac{\tan\theta_1}{\tan\theta_2} = \frac{\epsilon_1}{\epsilon_2}}$$

if $\epsilon_1 > \epsilon_2$, then $\theta_1 > \theta_2$

$$D_2^2 = D_{N2}^2 + D_{tan2}^2$$
$$= (D_1 \cos\theta_1)^2 + D^2_{tan2}$$

$$D_{tan2} = D_2 \sin\theta_2 = \frac{D_1 \sin\theta_1 \epsilon_2}{\epsilon_1}$$

$$D^2 = (D_1 \cos\theta_1)^2 + \left[D_1 \sin\theta_1 \frac{\epsilon_2}{\epsilon_1}\right]^2$$

$$D_2 = D_1 \sqrt{\cos^2\theta_1 + \left(\frac{\epsilon_2}{\epsilon_1}\right)^2 \sin^2\theta_1}$$

Similarly

$$E_2 = E_1 \sqrt{\sin^2\theta_1 + \left(\frac{\epsilon_1}{\epsilon_2}\right)^2 \cos^2\theta_1}$$

Relaxation Time

homogeneous → physical property doesnot vary from point to point

Non homogeneous → physical property vary from point to point.

linear → $\bar{D}$ is directly proportional to $\bar{E}$

non linear → $\bar{D}$ is not directly proportional to $\bar{E}$.

Relaxation time is defined as the time required by the charge density to decay to 36.8% of its initial value.

$$\tau = \text{Relaxation time} = \boxed{\frac{\epsilon}{\sigma}\ \text{sec}}$$

$\sigma \rightarrow$ conductivity

Capacitance

Two conducting surfaces carrying equal and opposite charges, seperated by a dielectric is called capacitance.

$$C = \frac{Q}{V}$$

$$1\ \text{Farad} = \frac{1\ \text{coulomb}}{1\ \text{volt}}$$

$$C = \frac{Q}{V} = \frac{\oint_s \bar{E} \cdot \epsilon \cdot d\bar{s}}{-\int_{+}^{+} \bar{E} \cdot d\bar{L}}\ F$$

Capacitors in series:

'n' capacitors in series

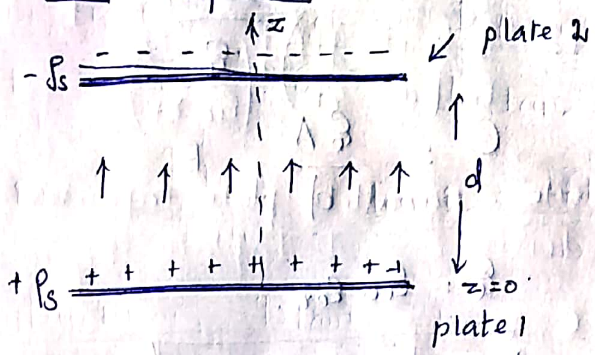
$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} \cdots + \frac{1}{C_n}.$$

Capacitors in parallel:

'n' capacitors in parallel

$$C_{eq} = C_1 + C_2 + \cdots C_n.$$

Parallel Plate Capacitor:



$$Q = \rho_s A \; C$$

$$E_1 = \frac{\rho_s}{2\epsilon_0}\bar{a}_N = \frac{\rho_s}{2\epsilon_0}\bar{a}_z \; V/m.$$

$$E_2 = \frac{-\rho_s}{2\epsilon_0}\bar{a}_N = -\frac{\rho_s}{2\epsilon_0}(-\bar{a}_z)$$

$$\bar{E} = \bar{E}_1 + \bar{E}_2 = \frac{\rho_s}{2\epsilon_0}\bar{a}_z + \frac{\rho_s}{2\epsilon_0}\bar{a}_z$$

$$\bar{E} = \frac{\rho_s}{\epsilon_0}\bar{a}_z$$

potential difference

$$V = -\int_{-}^{+} \bar{E}\cdot\bar{dL} = -\int_{upper}^{lower} \frac{\rho_s}{\epsilon_0}\bar{a}_z\cdot\bar{dL}$$

$$\bar{dL} = dx\,\bar{a}_x + dy\,\bar{a}_y + dz\,\bar{a}_z$$

$$V = -\int_{z=d}^{0} \frac{\rho_s}{\epsilon_0}\bar{a}_z \cdot \left[dx\,\bar{a}_x + dy\,\bar{a}_y + dz\,\bar{a}_z\right]$$

$$= -\int_{z=d}^{0} \frac{\rho_s}{\epsilon_0}\,dz$$

$$= -\frac{\rho_s}{\epsilon}\left[z\right]_d^0$$

$$= -\frac{\rho_s}{\epsilon}\left[-d\right]$$

$$\boxed{V = \frac{\rho_s d}{\epsilon} \quad V}$$

$$C = \frac{Q}{V} = \frac{\rho_s A}{\frac{\rho_s d}{\epsilon}}$$

$$C = \frac{\epsilon A}{d} \quad F$$

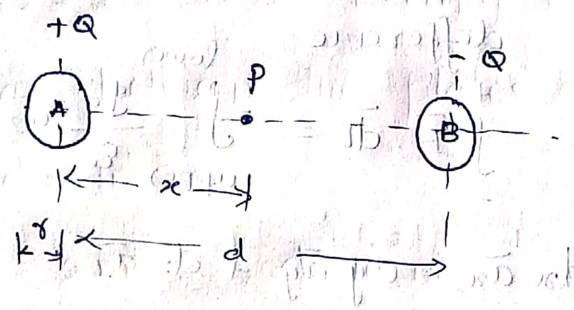
$$\epsilon = \epsilon_0 \epsilon_r .$$

$$\therefore \boxed{C = \frac{\epsilon_0 \epsilon_r A}{d}}$$

x ——— x

oblem ②

Find the capacitance between two parallel conductors. The radius of the conductor is 'r' replaced by a distance 'd' mtrs. Both wire are carrying the current in opposite direction.



The magnitude of the fields at point P due to both the conductors are

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$$|E_A| = \frac{Q}{2\pi \epsilon_0 x} , \quad |E_B| = \frac{Q}{2\pi \epsilon_0 (d-x)}$$

$$|E_P| = |E_A| + |E_B| = \frac{Q}{2\pi \epsilon_0} \left[\frac{1}{x} + \frac{1}{d-x} \right]$$

The potential difference between the two conductors,

$$V = - \int_{x=d-r}^{x=r} |E_P| \, dx$$

$$= - \int_{x=d-r}^{r} \frac{Q}{2\pi \epsilon_0} \left[\frac{1}{x} + \frac{1}{d-x} \right] dx$$

$$= \frac{-Q}{2\pi \epsilon_0} \Big[-\ln(d-x) + \ln x \Big]_{x=d-r}^{x=r}$$

$$= -\frac{Q}{2\pi \epsilon_0} \left[-\ln(d-r) + \ln r + \ln r - \ln (d-r) \right]$$

$$= \frac{-Q}{2\pi \epsilon_0} \left[2 \ln r - 2 \ln (d-r) \right]$$

$$V = \frac{Q}{\pi \epsilon_0} \left[\ln (d-r) - \ln r \right]$$

$$\boxed{V = \frac{Q}{\pi \epsilon_0} \ln \left[\frac{(d-r)}{r} \right]}$$

$$C = \frac{Q}{V}$$

$$\boxed{C = \frac{\pi \epsilon_0}{\ln \left(\frac{d-r}{r} \right)}}$$

__Capacitance in co axial cable__

$$C = \frac{2\pi \epsilon_0 L}{\ln (b/a)}$$

$L \rightarrow$ length of cable

$a \rightarrow$ inner radius

$b \rightarrow$ outer radius.

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Spherical Capacitor

$$C = \frac{4 \epsilon \pi}{\left[\frac{1}{a} - \frac{1}{b}\right]}$$

$a \rightarrow$ radius of inner sphere

$b \rightarrow$ radius of outer sphere.

Capacitance of Single Isolated Sphere

$$C = 4\pi \epsilon_0 a \quad F$$

Isolated sphere coated with Dielectric

$$C = \frac{4\pi}{\frac{1}{\epsilon_{01}}\left(\frac{1}{a} - \frac{1}{r_1}\right) + \left(\frac{1}{\epsilon_0 r_1}\right)} \quad F$$

Composite parallel plate capacitor

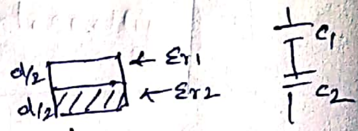
$$C = \frac{A}{\frac{d_1}{\epsilon_1} + \frac{d_2}{\epsilon_2} + \dots \frac{d_n}{\epsilon_n}}.$$

Dielectric Boundary normal to the plates

$$C = \frac{\epsilon_1 A_1}{d} + \frac{\epsilon_2 A_2}{d}$$

Problem ③

Find the capacitance of a parallel plate capacitor with dielectric $\epsilon_{r1} = 1.5$ and $\epsilon_{r2} = 3.5$ each occupy one half of the space between the plates of area $2\,m^2$ and $d = 10^{-3}\,m$.



$A = 2\,m^2$, $d = 10^{-3}\,m$

$d_1 = d_2 = d/2$

$$C_1 = \frac{\epsilon_1 A}{d_1} = \frac{\epsilon_0 \epsilon_{01} A}{d_1}$$

$$C_1 = \frac{8.854 \times 10^{-12} \times 1.5 \times 2}{10^{-3}/2}$$

$$\boxed{C_1 = 53.124\ nF}$$

$$C_2 = \frac{\varepsilon_2 A}{d_2} = \frac{\varepsilon_0 \varepsilon_{r2} A}{d_2}$$

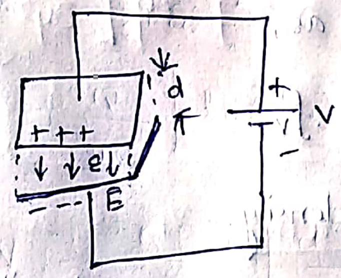
$$= \frac{8.854 \times 10^{-12} \times 3.5 \times 2}{10^{-3}/2}$$

$$\boxed{C_2 = 123.956\ nF}$$

$$C_{eq} = \frac{C_1 * C_2}{C_1 + C_2}$$

$$\boxed{C_{eq} = 37.19\ nF}$$

Energy stored in capacitor



Let $\overline{a_n}$ is the directional normal to the plates

$$\overline{E} = \frac{V}{d}\ \overline{a_N}$$

Energy stored $W_E = \frac{1}{2} \int_V \overline{D} \cdot \overline{E} \cdot dv$

$$= \frac{1}{2} \int_V \varepsilon \overline{E} \cdot \overline{E} \cdot dv$$

$$= \frac{1}{2} \int_V \varepsilon |\overline{E}|^2 dv \qquad \overline{E} = \frac{V}{d}$$

$$= \frac{1}{2} \varepsilon \frac{V^2}{d^2} \int_V dv$$

$$\int_V dv = \text{Volume} = A \times d.$$

$$\therefore W_E = \frac{1}{2} \epsilon_r \frac{V^2 A d}{d^2}$$

$$= \frac{1}{2} \frac{\epsilon_r A}{d} V^2$$

$$\boxed{W_E = \frac{1}{2} C V^2 \ J}$$

Energy density

$$W_E = \frac{1}{2} |\bar{D}||\bar{E}|. \ J/m^3.$$

Poisson's and Laplace Equations

From the Gauss's law in the point form, Poisson's equation can be derived. Consider the Gauss's law in the point form as,

$$\nabla \cdot \bar{D} = \rho_v$$

$\bar{D}$ = Flux density.

ρ_v - volume of charge density.

It is known that for a homogeneous, isotropic and linear medium, flux density and electric field intensity are directly proportional. Thus,

$$\bar{D} = \epsilon \bar{E}$$

$$\nabla \cdot \epsilon \bar{E} = \rho_v.$$

From the gradient relationship

$$\bar{E} = -\nabla V$$

Then $\nabla \cdot \epsilon_0 (-\nabla V) = \rho_v$

Taking $-\epsilon_0$ outside as constant

$$-\epsilon_0 [\nabla \cdot \nabla V] = \rho_v$$

$$\nabla \cdot \nabla V = -\frac{\rho_v}{\epsilon_0}$$

$$\nabla \cdot \nabla = \nabla^2$$

$$\boxed{\nabla^2 V = -\frac{\rho_v}{\epsilon_0}}$$

This equation is called Poisson's Equation.

For charge free region,

$$\nabla^2 V = 0$$

The ∇^2 operation is called the Laplacian of V.

∇^2 operation in different co-ordinate Systems

(i) In cartesian co-ordinate system.

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

(ii) In cylinderical co-ordinate systems

$$\nabla^2 V = \frac{1}{r} \frac{\partial}{\partial r} r \left(\frac{\partial V}{\partial r}\right) + \frac{1}{r^2} \left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial^2 V}{\partial z^2} = 0$$

(iii) In Spherical co-ordinate Systems:

$$\nabla^2 V = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 V}{\partial \phi^2} = 0$$

Problem④

Show that in cartesian co ordinates for any vector A, $\nabla \cdot (\nabla^2 A) = \nabla^2 (\nabla \cdot A)$

Solution)

$$\bar{A} = A_x \bar{a_x} + A_y \bar{a_y} + A_z \bar{a_z}$$

$$\nabla^2 \bar{A} = \nabla^2 A_x \cdot \bar{a_x} + \nabla^2 A_y \bar{a_y} + \nabla^2 A_z \bar{a_z}$$

L.H.S . $\nabla \cdot (\nabla^2 \bar{A}) = \nabla \cdot \left[\nabla^2 A_x \cdot \bar{a_x} + \nabla^2 A_y \bar{a_y} + \nabla^2 A_z \bar{a_z}\right]$

$$= \frac{\partial \nabla^2 A_x}{\partial x} + \frac{\partial \nabla^2 A_y}{\partial y} + \frac{\partial \nabla^2 A_z}{\partial z}$$

$$= \frac{\partial}{\partial x}\left[\frac{\partial^2 A_x}{\partial x^2}\right] + \frac{\partial}{\partial y}\left[\frac{\partial^2 A_y}{\partial y^2}\right] + \frac{\partial}{\partial z}\left[\frac{\partial^2 A_z}{\partial z^2}\right]$$

R.H.S . $\nabla^2 (\nabla \cdot \bar{A}) = \nabla^2 \left[\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right]$

$$= \frac{\partial^2}{\partial x^2}\left[\frac{\partial A_x}{\partial x}\right] + \frac{\partial^2}{\partial y^2}\left[\frac{\partial A_y}{\partial y}\right] + \frac{\partial^2}{\partial z^2}\left[\frac{\partial A_z}{\partial z}\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{\partial^2 A_x}{\partial x^2}\right] + \frac{\partial}{\partial y}\left[\frac{\partial^2 A_y}{\partial y^2}\right] + \frac{\partial}{\partial z}\left[\frac{\partial^2 A_z}{\partial z^2}\right]$$

LHS = RHS.

Uniqueness Theorem:-

Assume that the Laplace's equation has two solutions say V_1 and V_2. These solution must be satisfy Laplaces equation.

$$\nabla^2 V_1 = 0 \quad \& \quad \nabla^2 V_2 = 0$$

Both solution must be satisfy the boundry conditions.

At the boundry, the potential at

different points are same due to equipotential surface, then

$$V_1 = V_2$$

Let the difference between the two solutions

$$V_d = V_2 - V_1$$

Taking Laplace transform

$$\nabla^2 V_d = \nabla^2 (V_2 - V_1) = 0$$

$$\nabla^2 V_2 - \nabla^2 V_1 = 0$$

on the boundary $V_d = 0$

Now divergence theorem states that,

$$\int_V \nabla \cdot \bar{A} \, dv = \oint \bar{A} \cdot \bar{ds}$$

Let $\bar{A} = V_d \nabla V_d$

$$\nabla \cdot (\alpha \bar{B}) = \alpha (\nabla \cdot \bar{B}) + \bar{B} \cdot (\nabla \alpha)$$

Then $\nabla \cdot (V_d \nabla V_d)$ with $\alpha = V_d$ and $\nabla V_d = \bar{B}$

$$\therefore \nabla \cdot (V_d \nabla V_d) = V_d [\nabla \cdot \nabla V_d] + \nabla V_d \cdot [\nabla V_d]$$

$$\nabla \cdot \nabla = \nabla^2,$$

$$\nabla \cdot (V_d \nabla V_d) = V_d \nabla^2 V_d + \nabla V_d \cdot \nabla V_d$$

W.K.T $\nabla^2 V_d = 0$, then

$$\nabla \cdot (V_d \nabla V_d) = \nabla V_d \cdot \nabla V_d$$

and $\bar{A} = V_d \nabla V_d$, then

$$\nabla \cdot \bar{A} = \nabla V_d \cdot \nabla V_d$$

$$\int_V \nabla V_d \cdot \nabla V_d \, dv = \oint_S V_d \nabla V_d \cdot \bar{ds}$$

But $V_d = 0$ on boundry, then

$$\int_V \nabla V_d \cdot \nabla V_d \cdot dv = 0$$

This is volume integral to be evaluated on the volume enclosed by the boundry, then

$$\bar{C} \cdot \bar{C} = |\bar{C}|^2$$

$$\int_V |\nabla V_d|^2 dv = 0$$ as ∇V_d is vector.

Now integration can be zero under two condition

(i) The quantity under integral sign is zero

(ii) The quantity is positive in some regions and negative in other regions by equal amount and hence zero

$$|\nabla V_d|^2 = 0$$

$$\nabla V_d = 0$$

Gradient of $V_d = V_2 - V_1$ is zero

ie $$V_2 - V_1 = \text{constant} = 0$$

$$\therefore V_2 = V_1$$

If the solutions of Laplace's equation satisfies the boundry condition, then that solution is unique, by whatever method is obtained.

Problem ⑤

In spherical co ordinates $V = -25\ V$ on a conductor at $r = 2$ cm and $V = 150\ V$ at $r = 35$ cm. The space between the conductor is a dielectric of $\epsilon_r = 3.12$. Find the surface charge densities on the conductor.

Solution: For spherical system

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial V}{\partial r}\right] + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left[\sin\theta \frac{\partial V}{\partial \theta}\right] + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0$$

The voltage is a function of r only in spherical system, Hence.

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial V}{\partial r}\right]$$

Let, $r^2 \frac{\partial V}{\partial r} = A$

$$V = \int \frac{\partial V}{\partial r} = \int \frac{A}{r^2} = \int A$$

$$V = -\frac{A}{r} + B.$$

when $V = -25$ at $r = 2$ cm.

$$-25 = -\frac{A}{0.02} + B = -50A + B$$

when $V = 150V$ at $r = 35$ cm

$$150 = -\frac{A}{0.35} + B = -2.857A + B$$

Solving $A = 3.712$, $B = \ldots$

$$\therefore \boxed{V = -\frac{3.712}{r} + \ldots}$$

Problem ⑤

In spherical co ordinates $V = -25\,V$ on a conductor at $r = 2$ cm and $V = 150\,V$ at $r = 35$ cm The space between the conductor is a dielectric of $\epsilon_r = 3.12$. Find the surface charge densities on the conductor.

Solution: For spherical system

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial V}{\partial r}\right] + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left[\sin\theta \frac{\partial V}{\partial \theta}\right] + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0$$

The voltage is a function of r only in spherical system, Hence.

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial V}{\partial r}\right]$$

Let, $r^2 \frac{\partial V}{\partial r} = A$

$$V = \int \frac{\partial V}{\partial r} = \int \frac{A}{r^2} = \int A r^{-2}$$

$$V = -\frac{A}{r} + B.$$

when $V = -25$ at $r = 2$ cm,

$$-25 = -\frac{A}{0.02} + B = -50A + B \quad \text{--- ①}$$

when $V = 150V$ at $r = 35$ cm

$$150 = -\frac{A}{0.35} + B = -2.8571A + B \quad \text{--- ②}.$$

Solving $A = 3.712$, $B = 160.61$

$$\therefore \boxed{V = -\frac{3.712}{r} + 160.61}$$

$$\overline{E} = -\nabla V = -\frac{\partial V}{\partial r}\,\overline{a_r}$$

$$= -\frac{\partial}{\partial r}\left[-\frac{3.172}{r} + 160.61\right]\overline{a_r}$$

$$= -\frac{3.172}{r}\,\overline{a_r} \quad V/m$$

$$\overline{D} = \varepsilon_0 \varepsilon_r \overline{E} = 8.854 \times 10^{-12} \times 3.12 \times \left[-\frac{3.712}{r^2}\right]\overline{a_r}$$

$$\overline{D} = -\frac{0.103}{r^2}\,\overline{a_r} \quad nC/m^2$$

On a conductor surface, $D = \rho_s$

At $r = 0.02$ m, $\rho_s = -\frac{0.103}{(0.02)^2}$

$$\boxed{\rho_s = -257.5 \ nC/m^2}$$

At $r = 0.35$ m, $\rho_s = \frac{0.103}{(0.35)^2}$

$$\boxed{\rho_s = 0.841 \ nC/m^2}$$

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UNIT - 3

Magnetostatics

Lorentz Force:

The force exerted on a charged particle 'q' moving with velocity 'v' through an electric field 'E' and magnetic field 'B'. The entire electromagnetic force 'F' on the charged particle is called the Lorentz Force.

$$F = qE + qv \times B$$

Magnetic Field Intensity:

The force experienced by a unit north pole of one weber strength, when placed at the point ($\bar{H}$). unit (AT/m)

Magnetic flux density:

Magnetic flux crossing a unit area in a plane at right angles to the direction of flux is called Magnetic flux density. ($\bar{B}$)

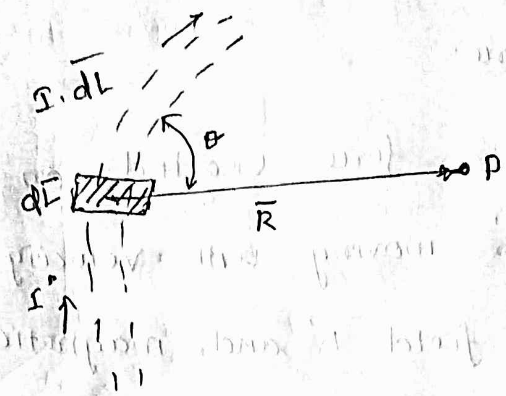
unit → Wb/m^2

Relation Between $\bar{B}$ & $\bar{H}$:

$$\bar{B} = \mu \bar{H} = \mu_0 \mu_r \bar{H}$$

non magnetic material $\mu_r = 1$

Biot Savart Law



Consider a conductor carrying a current I and steady state magnetic field produced around it.

dL → differential length.

IdL → differential current

θ → angle b/w differential current element & line joining point 'P'.

Biot Savart law states that

Magnetic field intensity dH produced at a point 'P' due to a differential current element IdL is

* proportional to the product of the current I and the differential length dL

* Sine of the angle between the element and the line joining point P to the element

* And inversely proportional to the square of the distance R between point P and the element.

$$d\bar{H} \propto \frac{I\,dl\,\sin\theta}{R^2}$$

$$d\bar{H} = \frac{K\,I\,dl\,\sin\theta}{R^2}$$

$K \rightarrow$ constant of proportionality.

$$K = \frac{1}{4\pi}$$

$$\therefore \quad d\bar{H} = \frac{I\,dL\,\sin\theta}{4\pi R^2}$$

$d\bar{L} \rightarrow$ magnitude of vector length $d\bar{L}$

$\bar{a}_R \rightarrow$ unit vector in the direction from differential element to point 'P'.

$$d\bar{L} \times a_R = dL\,|\bar{a}_R|\,\sin\theta$$

$$= dL\,\sin\theta$$

$$d\bar{H} = \frac{I\,d\bar{L} \times \bar{a}_R}{4\pi R^2} \quad A/m.$$

$$\bar{a}_R = \frac{\bar{R}}{|\bar{R}|} = \frac{\bar{R}}{R}$$

$$d\bar{H} = \frac{I\,d\bar{L} \times \bar{R}}{4\pi R^3} \quad A/m.$$

The total magnetic field intensity $\bar{H}$,

$$\bar{H} = \oint \frac{I\,d\bar{L} \times \bar{a}_R}{4\pi R^2} \quad A/m$$

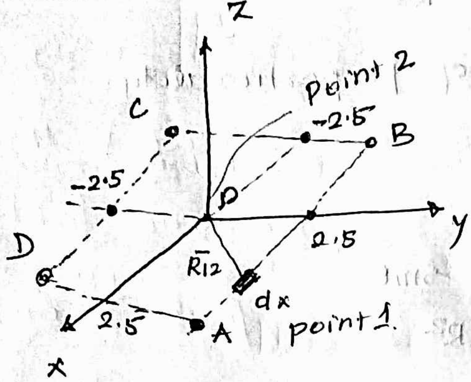
Biot Savart law can be expressed for surface element considering $\bar{K}\,ds$ while for volume current current considering $\bar{J}\,dv$

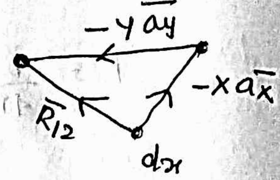
$$\bar{H} = \int_S \frac{\bar{K}(ds \times \bar{a}_R)}{4\pi R^2} \; A/m, \quad \bar{H} = \int_V \frac{\bar{J} \times \bar{a}_R\,dv}{4\pi R^2} \quad A/m.$$

This is ampere's law for the current element.

Pbm ①

Find the flux density at the centre 'P' of a Square of sides equal to 5 m and carrying 10 A of current.





Consider differential element dx along AB of square

$$d\bar{L} = dx\,\bar{a}_x \quad \text{——— ①}$$

$\bar{R}_{12}$ Joining differential element to point P

$$\bar{R}_{12} = -x\bar{a}_x - y\bar{a}_y$$

$$|\bar{R}_{12}| = \sqrt{x^2+y^2}$$

$$\bar{a}_{R12} = \frac{\bar{R}_{12}}{|\bar{R}_{12}|} = \frac{-x\bar{a}_x - y\bar{a}_y}{\sqrt{x^2+y^2}} \quad \text{— ②.}$$

$$d\bar{L} \times \bar{a}_{R12} = \begin{vmatrix} a_x & a_y & a_z \\ dx & 0 & 0 \\ -x & -y & 0 \end{vmatrix}$$

$$d\bar{L} \times \bar{a}_{R12} = -y\,dx\,\bar{a}_z$$

According to Biot Savart law,

$$d\bar{H} = \frac{I\,d\bar{L} \times \bar{a}_{R12}}{4\pi R_{12}^2}$$

$$= \frac{I\,(-y\,dx)\,\bar{a}_z}{4\pi\sqrt{x^2+y^2}\left(\sqrt{x^2+y^2}\right)^2}$$

$$d\bar{H} = \frac{10 \times (-2.5)\, dx\, \bar{a}_z}{4\pi\,(x^2 + 2.5^2)^{3/2}}$$

$$\bar{H} = \int_{x=2.5}^{-2.5} \frac{-25\, dx\, \bar{a}_z}{4\pi\,(x^2 + 2.5^2)^{3/2}}$$

$$= 2 \int_{x=2.5}^{0} \frac{-25\, dx\, a_z}{(x^2 + 2.5^2)^{3/2}}$$

put $x = 2.5 \tan\theta$, $dx = 2.5 \sec^2\theta\, d\theta$

Limits , $x = 2.5$, $\theta = 45^\circ$ & $x = 0$, $\theta = 0$.

$$\bar{H} = -\frac{25 \times 2}{4\pi} \int_{\theta=45^\circ}^{0^\circ} \frac{2.5 \sec^2\theta\, d\theta\, a_z}{(2.5)^3 \{1 + \tan^2\theta\}^{3/2}}$$

$$= -0.6366 \int_{45^\circ}^{0^\circ} \frac{1}{\sec\theta}\, d\theta\, \bar{a}_z$$

$$= -0.6366 \int_{45^\circ}^{0^\circ} \cos\theta\, d\theta\, \bar{a}_z$$

$$= -0.6366 \left[\sin\theta\right]_{45}^{0}$$

$$\bar{H} = 0.4501\, \bar{a}_z \ \ A/m$$

This $\bar{H}$ is segment AB of the square. All sides produces same $\bar{H}$ at point 'P'.

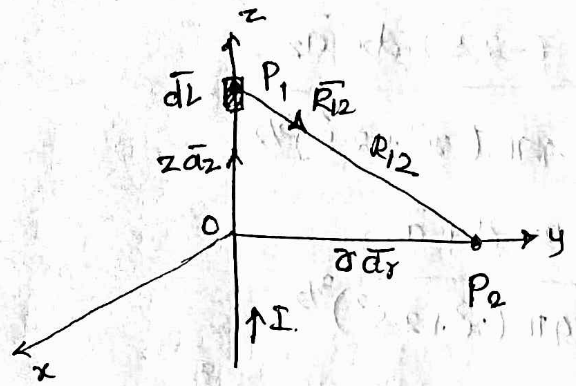
$$\bar{H}_{total} = 4\bar{H} = 4 \times 0.4501\, \bar{a}_z$$

$$\boxed{\bar{H}_{total} = 1.8\, \bar{a}_z \ A/m}$$

$\bar{H}$ Due to infinity long streight conductor

consider a small differential element at point 1 along the z-axis at a distance z

$$I\,d\bar{L} = I\,dz \qquad -(1)$$



Distance vector $\overline{R_{12}} = -z\,\bar{a_z} + r\,\bar{a_r}$

$$\overline{a_{R12}} = \frac{\overline{R_{12}}}{|\overline{R_{12}}|} = \frac{r\,\bar{a_r} - z\,\bar{a_z}}{\sqrt{r^2+z^2}}$$

$$\bar{dL} \times \overline{a_{R12}} = \begin{vmatrix} a_r & a_\varphi & a_z \\ 0 & 0 & dz \\ r & 0 & -z \end{vmatrix}$$

$$\bar{dL} \times \overline{a_{R12}} = r\,dz\,\overline{a_\varphi}$$

$$\therefore\ I\bar{dL} \times \overline{a_{R12}} = \frac{I.\,r\,dz\,\bar{a_\varphi}}{\sqrt{r^2+z^2}}$$

According to Biot savart law, dH at point 2 is

$$d\bar{H} = \frac{I\bar{dL} \times \overline{a_{R12}}}{4\pi R_{12}^2} = \frac{I\,r\,dz\,\overline{a_\varphi}}{4\pi\sqrt{r^2+z^2}\left(\sqrt{r^2+z^2}\right)^2}$$

$$\boxed{d\bar{H} = \frac{I\,r\,dz\,d\varphi}{4\pi\,(r^2+z^2)^{3/2}}}$$

The total $\bar{H}$ at P due to conductor of finite Length can be obtained by integrating dH over, $z = z_1$ to $z = z_2$.

$$\bar{H} = \int_{z_1}^{z_2} d\bar{H} = \int_{z_1}^{z_2} \frac{I\,r\,dz\,\overline{a_\varphi}}{4\pi\,(r^2+z^2)^{3/2}}$$

$z = r \tan\alpha \quad , \quad z^2 = r^2 \tan^2\alpha$

$dz = r \sec^2\alpha \, d\alpha .$

For $z = z_1 \quad , \quad z_1 = r \tan\alpha_1 ,$

$z = z_2 \quad , \quad z_2 = r \tan\alpha_2$

$\alpha_1 = \tan^{-1}(z_1/r)$ and $\alpha_2 = \tan^{-1}(z_2/r)$

$$\bar{H} = \int_{\alpha_1}^{\alpha_2} \frac{I r \, r \sec^2\alpha \, d\alpha \; \bar{a}_\varphi}{4\pi \left[r^2 + r^2 \tan^2\alpha \right]^{3/2}}$$

$$= \int_{\alpha_1}^{\alpha_2} \frac{I \, d\alpha \; \bar{a}_\varphi}{4\pi (\sec\alpha) . r}$$

$$= \frac{I}{4\pi r} \int_{\alpha_1}^{\alpha_2} \cos\alpha \, d\alpha \; \bar{a}_\varphi$$

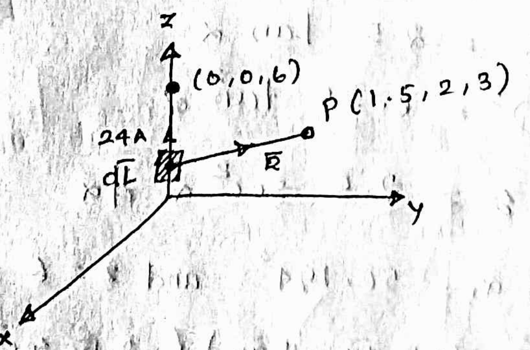
$$= \frac{I}{4\pi r} \left[\sin\alpha \right]_{\alpha_1}^{\alpha_2} \bar{a}_\varphi$$

$$\boxed{\bar{H} = \frac{I}{4\pi r} \left[\sin\alpha_2 - \sin\alpha_1 \right] . \bar{a}_\varphi \quad \text{Am}}$$

$$\bar{B} = \mu \bar{H} = \frac{\mu I}{4\pi r} \left[\sin\alpha_2 - \sin\alpha_1 \right] . \bar{a}_\varphi \quad \text{wb/m}^2$$

---

pbm 2 Find the magnetic intensity at $(1.5, 2, 3)$ due to a conductor carrying current of 24 A along z-axes extending from $z=0$ to $z=6$



$$\bar{dL} = dz\,\bar{a}_z$$

The point at which $\bar{dL}$ is considered is $(0,0,z)$

$$\bar{R} = 1.5\,\bar{a}_x + 2\,\bar{a}_y + (3-z)\,\bar{a}_z$$

$$|\bar{R}| = \sqrt{1.5^2 + 2^2 + (3-z)^2}$$

$$= \sqrt{6.25 + (3-z)^2}$$

$$\therefore \bar{a}_R = \frac{\bar{R}}{|\bar{R}|} = \frac{1.5\,\bar{a}_x + 2\,\bar{a}_y + (3-z)\,\bar{a}_z}{\sqrt{6.25 + (3-z)^2}}$$

$$I\bar{dL} \times \bar{a}_R = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ 0 & 0 & dz \\ 1.5 & 2 & 3-z \end{vmatrix} \times \frac{1}{\sqrt{6.25 + (3-z)^2}}$$

$$= \frac{24}{\sqrt{6.25 + (3-z)^2}} \left(1.5\,dz\,\bar{a}_y - 2\,dz\,\bar{a}_x\right)$$

According to Biot Savart law, $\bar{dH}$ at point P is

$$\bar{dH} = \frac{I\bar{dL} \times \bar{a}_R}{4\pi R^2} = \frac{24\left(1.5\,dz\,\bar{a}_y - 2\,dz\,\bar{a}_x\right)}{4\pi\sqrt{6.25 + (3-z)^2}\left[6.25 + (3-z)^2\right]}$$

$$= \frac{24}{4\pi} \frac{\left(1.5\,\bar{a}_y - 2\,\bar{a}_x\right)dz}{\left[6.25 + (3-z)^2\right]^{3/2}}$$

$$\bar{H} = \int_{z=0}^{z=6} \frac{24}{4\pi} \frac{\left(1.5\,\bar{a}_y - 2\,\bar{a}_x\right)dz}{\left[6.25 + (3-z)^2\right]^{3/2}}$$

use $\quad 3 - z = 2.5 \tan\alpha$

$$(3-z)^2 = 6.25 \tan^2\alpha$$

$$-dz = 2.5 \sec^2\alpha\, d\alpha$$

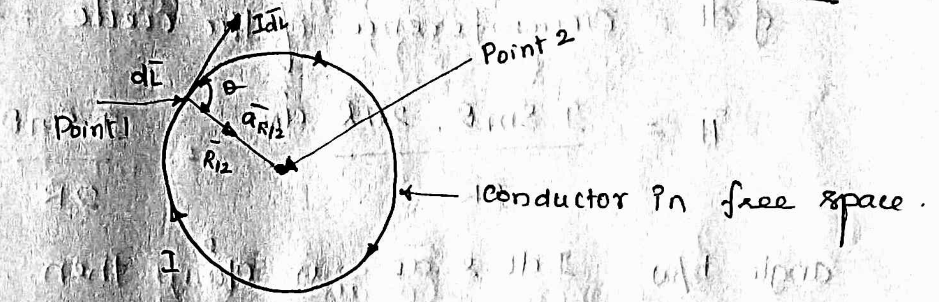
Limits $\quad z=0, \ \alpha_1 = 50.194^\circ$ and $z = 6, \ \alpha_2 = -50.194$

$$\bar{H} = \int_{\alpha_1}^{\alpha_2} \frac{24}{4\pi} \frac{\left(1.5\,\bar{a}_y - 2\,\bar{a}_x\right)\left(-2.5\sec^2\alpha\, d\alpha\right)}{(6.25)^{3/2}\left(1 + \tan^2\alpha\right)^{3/2}}$$

$$\bar{H} = \int_{\alpha_1}^{\alpha_2} \frac{24}{4\pi} (-2\,\bar{a}_x + 11.5\,\bar{a}_y) [\sin\alpha]_{50.194}^{-50.194}$$

$$\boxed{\bar{H} = -0.1987\,\bar{a}_x + 0.704\,\bar{a}_y \quad A/m}$$

$\bar{H}$ at the centre of a circular conductor



Consider the current carrying conductor arranged in a circular form. The $\bar{H}$ at the centre of the circular loop is to obtained. The conductor carries the direct current I.

Consider the differential length $\bar{dL}$ at a point 1. The direction of $\bar{dL}$ at point 1 is tangential to the circular conductor at point 1.

$\theta \rightarrow$ angle between $I\,\bar{dL}$ and $\bar{a}_{R12}$

$\bar{a}_{R12}$ – unit vector in the direction of $\bar{R}_{12}$

$\bar{R}_{12}$ – distance b/w diff. current element at point 1 to point 2.

$$I\,\bar{dL} \times \bar{a}_{R12} = I\,|\bar{dL}|\,|\bar{a}_{R12}| \sin\theta\,\bar{a}_N$$

$$= I\,dL \sin\theta\,\bar{a}_N$$

$\bar{a}_N \rightarrow$ unit vector normal to the plane $\bar{dL}$ & $\bar{a}_{R12}$

According to Biot savart law,

$$d\bar{H} = \frac{I\bar{dL} \times \bar{a}_{R12}}{4\pi R_{12}^2} = \frac{I dL \sin\theta \, \bar{a}_N}{4\pi R^2}$$

$$\bar{H} = \int d\bar{H} = \int \frac{I\bar{dL} \sin\theta \, \bar{a}_N}{4\pi R^2} = \frac{I \sin\theta \, \bar{a}_N}{4\pi R^2} \oint dL$$

$\oint dL$ = Circumference of the circle $= 2\pi R$

$$\bar{H} = \frac{I \sin\theta . 2\pi R \, \bar{a}_N}{2R} = \frac{I \sin\theta}{2R} \bar{a}_N$$

angle b/w $I dL$ & $\bar{a}_R$ is 90°, then.

$$\bar{H} = \frac{I \sin 90^\circ}{2R} \bar{a}_N$$

$$\boxed{\bar{H} = \frac{I}{2R} \bar{a}_N \quad A/m}$$

$\bar{a}_N = \bar{a}_z$ if the circular loop is placed in xy plane.

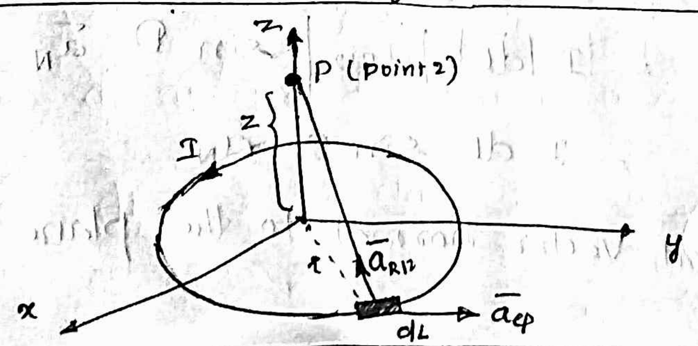
$$\bar{B} = \mu_0 \bar{H}$$

$$\therefore \bar{B} = \frac{\mu_0 I}{2R} \bar{a}_N \quad Wb/m^2$$

If the circular coil has N turns then $\bar{H}$ at its centre is

$$\bar{H} = \frac{NI}{2R} \bar{a}_N \quad \text{and} \quad \bar{B} = \frac{\mu_0 N I}{2R} \bar{a}_N$$

$\bar{H}$ on the axis of a circular loop:



In the cylindrical co ordinate systems,

$$\bar{dL} = dr\,\bar{a_r} + r\,d\varphi\,\bar{a_\varphi} + dz\,\bar{a_z}$$

$d\bar{L}$ is the plane for which r is constant and $z = 0 =$ constant plane. The $I\bar{dL}$ is tangential at point 1 in $\bar{a_\varphi}$ direction

$$I\,d\bar{L} = I\,r\,d\varphi\,\bar{a_\varphi}$$

The unit vector, $\bar{a}_{R12} = \dfrac{\bar{R_{12}}}{|\bar{R_{12}}|}$

$$\bar{R_{12}} = -r\,\bar{a_r} + z\,\bar{a_z}$$

$$|\bar{R_{12}}| = \sqrt{(-r)^2 + (z)^2}$$

$$= \sqrt{r^2 + z^2}$$

$$\bar{a}_{R12} = \frac{-r\,\bar{a}_r + z\,\bar{a}_z}{\sqrt{r^2 + z^2}}$$

Now $\quad d\bar{L} \times \bar{a}_{R12} = \begin{vmatrix} \bar{a_r} & \bar{a_\varphi} & \bar{a_z} \\ 0 & r\,d\varphi & 0 \\ -r & 0 & z \end{vmatrix}$

$$= zr\,d\varphi\,\bar{a_r} + r^2\,d\varphi\,\bar{a_z}$$

According to Biot Savart law,

$$d\bar{H} = \frac{I\bar{dL} \times \bar{a}_{R12}}{4\pi R_{12}^2} = \frac{I\left[zr\,d\varphi\,\bar{a_r} + r^2\,d\varphi\,\bar{a_z}\right]}{4\pi\sqrt{r^2+z^2}\left[\sqrt{r^2+z^2}\right]^2}$$

$d\bar{H}$ consists of two components $\bar{a_r}$ and $\bar{a_z}$, due to radial symmetry all $\bar{a_r}$ components are going cancelled. So $\bar{H}$ exists only along the axis in $\bar{a_z}$ direction.

$$\bar{H} = \frac{I}{4\pi}\int_{\varphi=0}^{2\pi} \frac{r^2\,d\varphi}{(r^2+z^2)^{3/2}}\cdot\bar{a_z}$$

$$\bar{H} = \frac{I r^2 \bar{a_z}}{4\pi (r^2+z^2)^{3/2}} \int_{\varphi=0}^{2\pi} d\varphi$$

$$= \frac{I r^2 \bar{a_z} [\varphi]_0^{2\pi}}{4\pi (r^2+z^2)^{3/2}}$$

$$\boxed{\bar{H} = \frac{I r^2}{2(r^2+z^2)^{3/2}} \bar{a_z} \quad A/m}$$

x x

Ampere's Circuital Law

The line integral of magnetic field intensity $\bar{H}$ around a closed path is exactly equal to the direct current enclosed by that path.

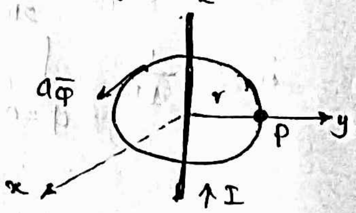
$$\oint \bar{H} \cdot \bar{dL} = I$$

Proof:

consider

- long straight conductor carrying direct current I placed along z axis.
- closed circular path of radius r which encloses streight conductor carrying direct current I.
- Point 'P' Ly distance 'r' from the conductor
- $d\bar{L}$ at poin 'P' which is in $\bar{a_\varphi}$.

$$\bar{dL} = r \, d\varphi \, \bar{a_\varphi}$$



while $\bar{H}$ obtained at point 'P'

$$\bar{H} = \frac{I}{2\pi r} \bar{a_\phi}$$

$$\bar{H} \cdot \bar{dL} = \frac{I}{2\pi r} \bar{a_\phi} \cdot r\, d\phi\, \bar{a_\phi}$$

$$= \frac{I}{2\pi \cancel{r}} \cdot \cancel{r} \cdot d\phi$$

$$\bar{H} \cdot \bar{dL} = \frac{I}{2\pi} d\phi$$

By integrating,

$$\oint \bar{H} \cdot \bar{dL} = \int_{\phi=0}^{2\pi} \frac{I}{2\pi} d\phi$$

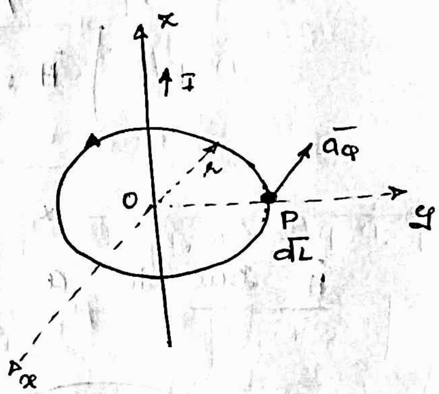
$$= \frac{I}{2\pi} [\phi]_0^{2\pi}$$

$$\boxed{\oint \bar{H} \cdot \bar{dL} = I}$$ Hence proved

Applications of Amphere's circuit law

1) $\bar{H}$ due to infinitely long straight conductor.

Consider an infinitely long straight conductor placed along z-axis, carrying a direct current I.



Consider point 'P' on the closed path at which $\bar{H}$ is to obtained. The radius of the path is 'r' and hence P is at a perpendicular distance 'r' from the conductor.

The magnitude of $\bar{H}$ depends on r and the direction is always tangential to the closed path $\bar{a_\phi}$

$\bar{H}$ has only component $\bar{a_\varphi}$ direction (H_φ).

consider $d\bar{L}$ at point P and in cylinderical co ordinates it is $r\, d\varphi$ in a_φ direction.

$$\bar{H} = H_\varphi \, \bar{a_\varphi} \quad \& \quad d\bar{L} = r\, d\varphi \, \bar{a_\varphi}$$

$$\bar{H} \cdot d\bar{L} = H_\varphi \bar{a}_\varphi \cdot r\, d\varphi \, \bar{a_\varphi}$$

$$= H_\varphi \cdot r \cdot d\varphi$$

According to Ampere's circuital law,

$$\oint \bar{H} \cdot d\bar{L} = I$$

$$\int_0^{2\pi} H_\varphi \cdot r\, d\varphi = I$$

$$H_\varphi \cdot r \int_0^{2\pi} d\varphi = I$$

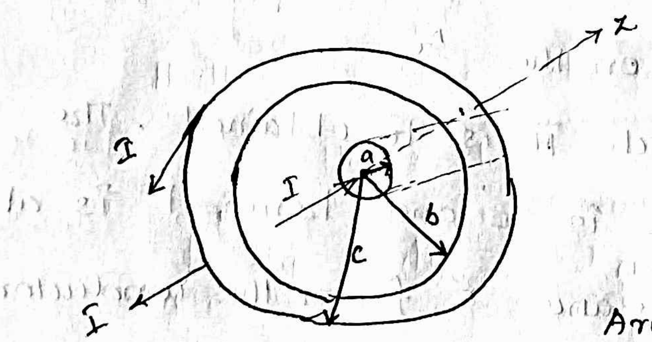
$$H_\varphi \cdot r \cdot (2\pi) = I$$

$$\bar{H_\varphi} = \frac{I}{2\pi r}$$

$\bar{H}$ at point 'p' is given by

$$\boxed{\bar{H} = H_\varphi \cdot a_\varphi = \frac{I}{2\pi r} \cdot \bar{a_\varphi}}$$

$\bar{H}$ Due to a co axial cable :-



Region 1 :-

- inner conductor

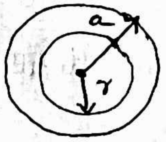
$r < a$

Area $= \pi r^2 \ m^2$

The total current flowing is I through area πa^2,

$$I' = \frac{\pi r^2}{\pi a^2} I$$

$$\boxed{I' = \frac{r^2}{a^2} I}$$



inner conductor

$$\bar{H} = H_\varphi \, \bar{a}_\varphi$$

$$\bar{dL} = r \, d\varphi \, \bar{a}_\varphi$$

$$\bar{H} \cdot \bar{dL} = \bar{H} \cdot \bar{dL} = H_\varphi \bar{a}_\varphi \cdot r \, d\varphi \, \bar{a}_\varphi$$

$$\bar{H} \cdot \bar{dL} = H_\varphi \cdot r \cdot d\varphi$$

According to Ampere's circuital law

$$\oint \bar{H} \cdot \bar{dL} = I'$$

$$\oint H_\varphi \cdot r \cdot d\varphi = \frac{r^2}{a^2} I$$

$$\int_{\varphi=0}^{2\pi} H_\varphi \, r \cdot d\varphi = \frac{r^2}{a^2} \cdot I$$

$$H_\varphi \cdot r \, [2\pi] = \frac{r^2}{a^2} \cdot I$$

$$H_\varphi = \frac{r^2}{2\pi r a^2} I = \frac{r}{2\pi a^2} I$$

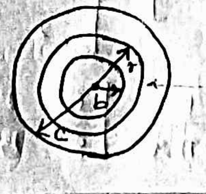
$$\therefore \boxed{\bar{H} = \frac{I r}{2\pi a^2} \bar{a}_\varphi \quad A/m}$$

Region 2: with $a < r < b$

consider a circular path which encloses the inner conductor carrying direct current I. This is the case of infinitely long conductor along z-axis

$$\therefore \boxed{\bar{H} = \frac{I}{2\pi r} \bar{a}_\varphi \quad A/m}$$

Region 3: with in outer conductor $b < r < c$



closed path

The total current $-I$ is flowing through the cross section $\pi(c^2 - b^2)$ while the closed path encloses the cross section $\pi(r^2 - b^2)$

The current enclosed by the closed path of outer conductor,

$$I' = \frac{\pi(r^2 - b^2)}{\pi(c^2 - b^2)} \times -I$$

$$I' = -\frac{r^2 - b^2}{c^2 - b^2} \cdot I$$

$I'' = I =$ current in inner conductor enclosed

Total current enclosed by the closed path

$$I_{enc} = I' + I'' = -\frac{r^2 - b^2}{c^2 - b^2} I + I$$

$$\boxed{I_{enc} = I\left[\frac{c^2 - r^2}{c^2 - b^2}\right]}$$

According to ampere circuit law,

$$\oint \bar{H} \cdot \bar{dL} = I_{enc}$$

$\bar{H}$ is again in $\bar{a}_\varphi$ direction only

$$\bar{H} = H_\varphi a_\varphi \quad \text{and} \quad \bar{dL} = r\, d\varphi\, \bar{a}_\varphi$$

$$\bar{H} \cdot \bar{dL} = H_\varphi \bar{a}_\varphi \cdot r\, d\varphi\, \bar{a}_\varphi$$

$$= H_\varphi \cdot r \cdot d\varphi$$

$$\int_{\varphi=0}^{2\pi} H_\varphi \cdot r \cdot d\varphi = I_{enc}$$

$$H_\varphi \cdot r \cdot [2\pi] = I\left[\frac{c^2 - r^2}{c^2 - b^2}\right]$$

$$\bar{H} = H_\varphi \cdot \bar{a}_\varphi = \frac{1}{2\pi r}\left[\frac{c^2 - r^2}{c^2 - b^2}\right] \cdot \bar{a}_\varphi \quad A/m$$

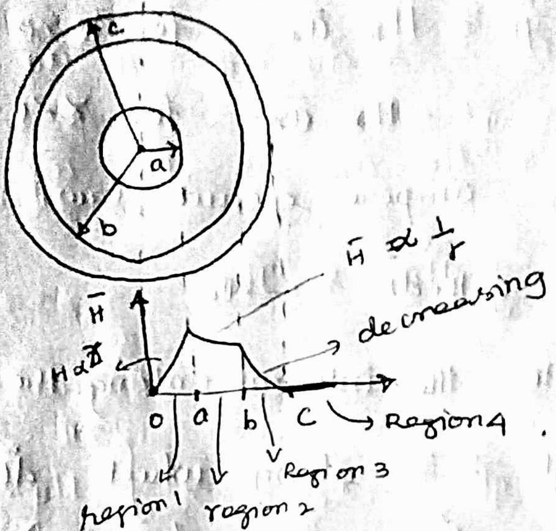
Region 4 : outside the cable , $r > c$

$I_{enc} = +I - I = 0\ A$

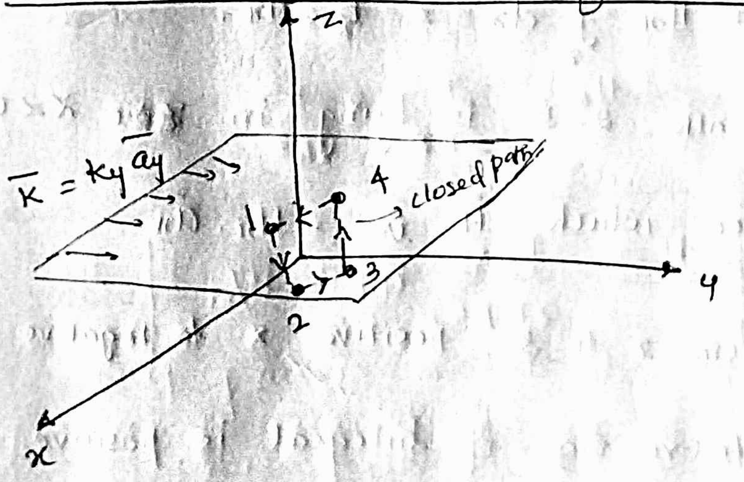
$\oint \bar{H} \cdot \bar{dL} = 0$

$\bar{H} = 0\ A/m$

The magnetic field doesnot exists outside the cable.



$\bar{H}$ due to infinite sheet of current



consider an infinite sheet of current in the $z = 0$ plane. Surface current density $\bar{K}$.

$$\bar{K} = k_y \bar{a_y}$$

consider a closed path, 1-2-3-4. width of the path is 'b', length of the path is 'a'.

The current flowing across the distance b is given by $K_y.b$. As current is flowing in y direction, $\bar{H}$ can not have component in y direction.

$$\bar{H} = H_x \, \bar{a_x} \quad \text{for } x > 0$$
$$= -H_x \, \bar{a_x} \quad \text{for } x < 0$$

Apply ampere circuit law,

$$\oint \bar{H} \cdot d\bar{L} = I_{enc}$$

Evaluate the integral along the path 1-2-34-1

For path 2-3 along which $d\bar{L} = dx \, \bar{a_x}$

$$\int_2^3 \bar{H} \cdot d\bar{L} = \int_2^3 (-H_x \, \bar{a_x}) \cdot (dx \cdot \bar{a_x})$$

$$= H_x \int_2^3 dx = b \cdot H_x .$$

The path 2-3 is lying in $x < 0$ region for which $\bar{H}$ is $-H_x \cdot \bar{a_x}$

Limits from 2 to 3, positive x to negative x hence effective sign of integral is positive.

Consider path 4-1 along which $d\bar{L} = dx \, \bar{a_x}$ and it is in the region $x > 0$, hence

$$\bar{H} = H_x \, \bar{a_x}$$

$$\int_4^1 \bar{H} \cdot d\bar{L} = \int_4^1 (H_x \, \bar{a_x}) \cdot (dx \, \bar{a_x}) = H_x \int_4^1 dx$$

$$\int_4^1 \bar{H} \cdot d\bar{L} = b \, H_x$$

$\oint \bar{H} \cdot \bar{dL} = b\, H_x + b\, H_x$

$\oint \bar{H} \cdot \bar{dL} = 2b\, H_x$

Equating to I_{enc}

$$2b\, H_x = K_y\, b$$

$$\boxed{H_x = \frac{1}{2} K_y}$$

$$\bar{H} = \frac{1}{2} K_y\, \bar{a}_x \quad \text{for } x > 0$$
$$= -\frac{1}{2} K_y\, \bar{a}_x \quad \text{for } z < 0$$

For an infinite sheet of current density $\bar{K}$ A/m, we can write

$$\boxed{\bar{H} = \frac{1}{2} \bar{K} \times \bar{a}_N}$$

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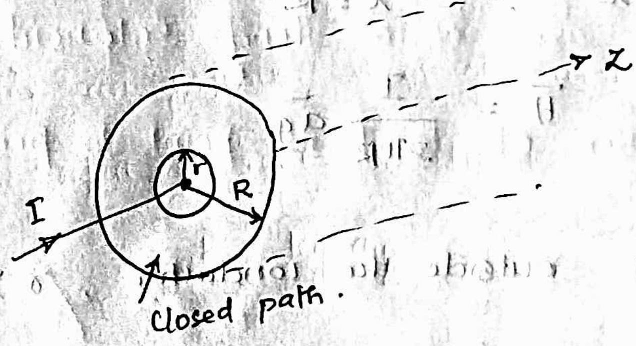
pbm ③ Obtain the expression for $\bar{H}$ in all the regions if a cylindrical conductor carries a direct current I and its radius is 'R' m. Plot the variation of H against the distance r from the centre of the conductor.



Region 1 :- within the conductor, $r < R$

As current I flows uniformly, its flow across the cross sectional area of πR^2

Hence current enclosed by the path

$$I_{enc} = I \frac{\pi r^2}{\pi R^2} = I \times \frac{r^2}{R^2}$$

$\bar{H}$ has only $\bar{a_\varphi}$ component

$$\bar{H} = H_\varphi \bar{a_\varphi}$$

$$\bar{dL} = r\, d\varphi\, \bar{a_\varphi} \quad \text{in } a_\varphi \text{ direction.}$$

$$\bar{H} \cdot \bar{dL} = H_\varphi \cdot \bar{a_\varphi} \cdot r \cdot d\varphi \cdot \bar{a_\varphi}$$

$$= H_\varphi \cdot r \cdot d\varphi$$

According to ampere's circuit law,

$$\oint \bar{H} \cdot \bar{dL} = I_{enc} \Rightarrow \int_{\varphi=0}^{2\pi} H_\varphi \cdot r \cdot d\varphi = I \frac{r^2}{R^2}$$

$$H_\varphi\, r \cdot [2\pi] = I \frac{r^2}{R^2}$$

$$H_\varphi = \frac{I}{2\pi r} \times \frac{r^2}{R^2}$$

$$\boxed{\bar{H} = \frac{I \cdot r}{2\pi R^2} \bar{a_\varphi} \quad H/m}$$

$\bar{H} \propto r$, $\therefore r = R$

$$\therefore \bar{H} = \frac{I}{2\pi R} \bar{a_\varphi}$$

Region 2 :- outside the conductor , $r > R$

$$\bar{H} = \frac{I}{2\pi r} \bar{a_\varphi} \quad \text{for } r > R$$

$$\bar{H} \propto \frac{1}{r}$$

Magnetic Flux and Flux Density

$$\bar{B} = \mu \cdot \bar{H}$$

for free space, $\mu = \mu_0 = 4\pi \times 10^{-7}$ H/m.

$\bar{B} = \mu_0 \bar{H}$ → for free space.

If the flux passing through the unit area is not exactly at right angles to the plane consisting the area but making some angle with the plane then the flux passing the area is given by

$$\phi = \int_S \bar{B} \cdot \bar{ds}$$

The integral $\bar{B} \cdot \bar{ds}$ evaluated over a closed surface is always zero.

$$\oint \bar{B} \cdot \bar{ds} = 0$$

This is called law of conservation of magnetic flux (or) Gauss law of integral form of magnetic fields.

Apply divergence Theorem,

$$\oint B \cdot \bar{ds} = \int \nabla \cdot \bar{B} \, dv = 0.$$

dv → volume enclosed by the closed surface.

As $dv \neq 0$, then

$$\nabla \cdot \bar{B} = 0$$

(ie) The divergence of magnetic flux density is always zero.

Maxwell's Equation for Static Electromagnetic Fields

Differential or point form

$\nabla \cdot \bar{D} = \rho_v$ — Gauss's law

$\nabla \times \bar{E} = 0$ — Conservation of Electric field

$\nabla \times \bar{H} = \bar{J}$ — Ampere's Circuit law

$\nabla \cdot \bar{B} = 0$ — conservation of magnetic flux

Integral form:-

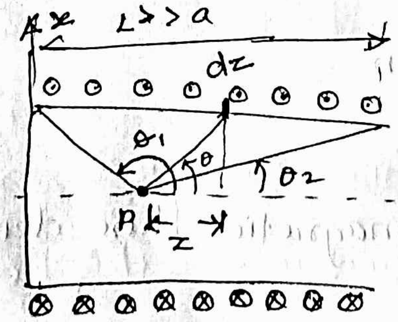
$$\oint \bar{D} \cdot d\bar{s} = \int_V \rho_v \, dv = Q$$

$$\oint \bar{E} \cdot d\bar{l} = 0$$

$$\oint \bar{H} \cdot d\bar{L} = \int_S \bar{J} \cdot d\bar{s} = I$$

$$\oint \bar{B} \cdot d\bar{s} = 0.$$

Pbm (4) Drive a general expression for the magnetic flux density B at any point along the axis of a long Solenoid. Sketch the variation of B from point to point along the axis.



$\rightarrow \sin\theta = a/\sqrt{a^2+z^2}$

$\rightarrow \tan\theta = \frac{a}{z}$

The solinoid made up of turns which are arranged in circular loops. ~~sol~~. Thus circular to dz produces a magnetic field at point 'P' is at a distance z on ~~the~~ axes.

Let current through solinoid is I ampere

$$dH = \frac{I a^2 \, dL}{2\left[a^2+z^2\right]^{3/2}}$$

circular loop $r = a$. $\frac{N}{L} \rightarrow$ no of turns per unit length.

$$dl = \frac{N}{L} \cdot dz$$

$$dH = \frac{I a^2 \, N \, dz}{2L \left[a^2+z^2\right]^{3/2}}$$

$\tan\theta = a/z$, $z = \frac{a}{\tan\theta} = a \cot\theta$

$$\boxed{z = a \cot \frac{1}{\theta}}$$

$$dz = -a \operatorname{cosec}^2\theta \, d\theta = \frac{a}{\sin^2\theta} = \frac{-a\sin\theta}{\sin^3\theta}.$$

$\sin\theta = \frac{a}{a^2+z^2}$ ie $\sin^3\theta = \frac{a^3}{(a^2+z^2)^{3/2}}$

$$dz = \frac{-\sin\theta \, (a^2+z^2)^{3/2}}{a^3}$$

$$\boxed{dz = \frac{-(a^2+z^2)^{3/2} \sin\theta \cdot d\theta}{a^2}}$$

$$\therefore \; d\bar{H} = \frac{I a^2 N}{2L} \times \frac{-(a^2+z^2)^{3/2} \sin\theta \, d\theta}{a^2(a^2+z^2)^{3/2}}$$

$$\boxed{d\bar{H} = \frac{-IN}{2L} \sin\theta \, d\theta}$$

$$\vec{H} = -\frac{NI}{2L} \int_{\theta_1}^{\theta_2} \sin\theta \, d\theta.$$

$$= \frac{NI}{2L} \left[\cos\theta_2 - \cos\theta_1\right] a_\varphi$$

$$\bar{B} = \frac{\mu N I}{2L} \left[\cos\theta_2 - \cos\theta_1\right] a_z.$$

$L >>> a$, $\hat{\theta}_2 = 0$, $\theta_1 = 180$

$$\therefore \; \bar{B} = \frac{\mu I N}{2L} \times (\cos 90^\circ - \cos 180^\circ) \bar{a}_z$$

$$\boxed{\bar{B} = \frac{\mu N I}{2L} a_z} \; wb/m^2$$

when $\theta_2 = 90$, $\theta_3 = \theta_i$

$$\bar{B} = \frac{\mu N I}{2L} \left[\cos 90 - \cos 180\right]$$

$$\bar{a}_z = \frac{\mu N I}{2L} \bar{a}_z$$

Poissons Equations for magnetic field

$$\nabla^2 \bar{A} = -\mu_0 \bar{J}$$

A due to differential current element

$$\bar{A} = \oint \frac{\mu_0 I \, d\bar{L}}{4\pi R} \quad wb/m$$

(or)

$$\bar{A} = \oint \frac{\mu_0 \bar{K} \, d\bar{s}}{4\pi R} \quad wb/m.$$

(or)

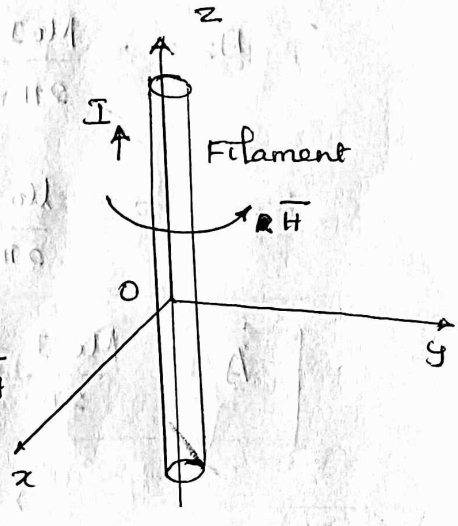
$$\bar{A} = \oint \frac{\mu_0 \bar{J} \, dv}{4\pi R} \quad wb/m.$$

---

pbm (5)

Obtain an expression for magnetic vector potential in the region surrounding an infinitely long straight filamentry current I.

Consider an infinitely long filament is given by



$$\bar{H} = \frac{I}{2\pi r} \bar{a_\varphi}$$

If it is placed in $\bar{B} = \mu_0 \bar{H}$

$$\bar{B} = \frac{\mu_0 I}{2\pi r} \bar{a_\varphi}$$

Assume cylinderical Co-ordinate system

$$\bar{B} = \nabla \times \bar{A}$$

$$\frac{\mu_0 I}{2\pi r} a_\varphi = \left[\frac{1}{r}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right] \bar{a_r} + \left[\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right] \bar{a_\varphi} + \frac{1}{r}\left[\frac{\partial (r A_\varphi)}{\partial r} - \frac{\partial A_r}{\partial \varphi}\right] \bar{a_z}$$

Equate the co-efficient of $\bar{a}_\varphi$.

$$\bar{B} = \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} = \frac{\mu_0 I}{2\pi r}$$

and $$\frac{\partial A_r}{\partial z} = 0$$

$$\frac{\partial A_z}{\partial r} = -\frac{\mu_0 I}{2\pi r}$$

$$A_z = -\int \frac{\mu_0 I}{2\pi r} dr + C_1$$

$$A_z = -\frac{\mu_0 I}{2\pi} \ln[r] + C_1$$

Find c, $A_z = 0$, $r = r_0$.

$$0 = -\frac{\mu_0 I}{2\pi} \ln[r_0] + C_1$$

$$C_1 = \frac{\mu_0 I}{2\pi r} \ln[r_0]$$

$$\therefore A_z = -\frac{\mu_0 I}{2\pi} \ln[r] + \frac{\mu_0 I}{2\pi} \ln[r_0]$$

$$= \frac{\mu_0 I}{2\pi} \ln\left[\frac{r_0}{r}\right]$$

$$\boxed{\bar{A} = \frac{\mu_0 I}{2\pi} \ln\left[\frac{r_0}{r}\right] \cdot \bar{a}_z}$$

Unit - 4 Electrodynamic Fields

Faraday's law:

The electromotive force (e.m.f) induced in a closed path is proportional to rate of change of magnetic flux enclosed by the closed path.

$$e = -N \frac{d\varphi}{dt}$$

$N \rightarrow$ no of turns in a circuit.

$e \rightarrow$ induced emf.

When $N = 1$ (single turn circuit)

$$e = -\frac{d\varphi}{dt}$$

Lenz's law:

The direction of induced emf is such that it opposes the cause producing it. (i.e) changes in the magnetic flux.

W.K.T

$$e = -N \frac{d\varphi}{dt}$$

in scalar form

$$e = \oint \bar{E} \cdot d\bar{L}$$

magnetic flux passing through specified area

$$\varphi = \int_S \bar{B} \cdot d\bar{s}$$

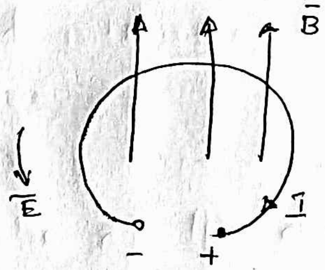
$$e = -\frac{d}{dt}\int_S \bar{B}.d\bar{s}$$

When an emf is induced in a stationary closed path due to time varying $\bar{B}$ field, the emf is called statically induced emf. $\boxed{\text{transformer emf}}$

when the emf is induced in a time varying closed path due to a static $\bar{B}$ field, the emf is called dynamically induced emf. $\boxed{\text{motional emf}}$

Statically induced emf:-

The closed circuit in which emf is induced in stationary and magnetic flux is sinusoidally varying with time.



$$\oint \bar{E}.d\bar{l} = -\int_S \frac{\partial \bar{B}}{\partial t} \cdot d\bar{s}$$

It is similar to transformer action and emf is called transformer emf.

$$\oint (\nabla \times \bar{E}).d\bar{s} = -\int_S \frac{\partial \bar{B}}{\partial t}.d\bar{s}$$

$$\nabla \times \bar{E}.d\bar{s} = -\frac{\partial \bar{B}}{\partial t}.d\bar{s}$$

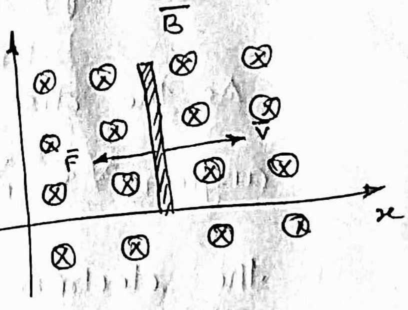
$$\therefore \boxed{\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}}$$

Based on maxwell's eqn

$$\oint \bar{E}.d\bar{l} = 0 \quad , \quad \nabla \times \bar{E} = 0.$$

Dynamically induced emf

Magnetic field is stationary, constant not varying with time while the closed circuit is revolved to get the relative motion between them. This is generator action, hence the induced emf is called motional or generator emf.



Force on a charge is given by

$$\bar{F} = Q\bar{V} \times \bar{B}$$

Electric field intensity

$$\bar{E} = \frac{\bar{F}}{Q} = \bar{V} \times \bar{B}$$

$$\oint \bar{E} \cdot d\bar{L} = \oint (\bar{V} \times \bar{B}) \cdot d\bar{L}$$

If the directions of velocity $\bar{V}$ with which conductor is moving and the magnetic field $\bar{B}$ are mutually perpendicular to each other, then the induced emf is given by

$$e = Blv \sin 90^\circ$$

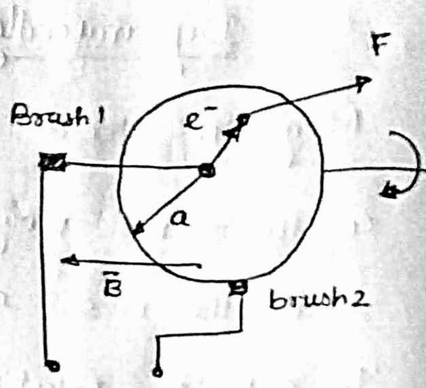
$$\boxed{e = BLV}$$ $l \rightarrow$ length of straight conductor.

Moving closed path in a time varying $\bar{B}$ field

Total induced emf = Transformer emf + Motional emf

$$\oint \bar{E} \cdot d\bar{L} = \int \frac{\partial \bar{B}}{\partial t} \cdot d\bar{s} + \oint (\bar{V} \times \bar{B}) \cdot d\bar{L}$$

Faraday's Disc Generator:



As the disc rotates at an angular velocity ω rad/sec. the electron moves at a velocity which is given by

$$V = \omega r \text{ m/s}$$

The force exerted on electron is given by

$$\bar{F} = q(\bar{V} \times \bar{B})$$

Electric field intensity

$$E = \frac{\bar{F}}{e} (\bar{V} \times \bar{B})$$

Magnitude of the electric field intensity

$$E = |\bar{E}| = (\omega r) B$$

Emf produced between centre of the disc and rim of the disc is given by

$$e = \int_0^a E\, dr = \int_0^a \omega r B\, dr$$

$$= \omega B \left[\frac{r^2}{2}\right]_0^a$$

$$\boxed{e = \frac{1}{2}\omega B a^2 \text{ V}}$$

A conducting loop of radius 10 cm lies in the $z=0$ plane. The associated $\bar{B} = 10 \sin(120\pi t)\, \bar{a}_z$ mWb/m^2 calculate voltage induced in the loop.

Soln:

$$\phi = \int_S \bar{B} \cdot d\bar{s}$$

$$\phi = \int_{\phi=0}^{2\pi} \int_{r=0}^{0.1} \left[10 \times 10^{-2} \sin(120\pi t) \vec{a_z} \right] \cdot (r \, dr \, d\phi) \, \vec{a_z}$$

$$= \left[\phi\right]_0^{2\pi} \left[\frac{r^2}{2} \right]_0^{0.1} 10^{-2} \sin(120\pi t)$$

$$= 2\pi \times \frac{0.1^2}{2} \times 10^{-2} \sin 120\pi t$$

$$\phi = 0.314 \times 10^{-3} \sin(120\pi t) \text{ Wb}$$

Emf induced is given by

$$e = -\frac{d\phi}{dt} = -\frac{d}{dt}(0.314 \times 10^{-3} \sin 120\pi t)$$

$$= -0.314 \times 10^{-3} (120 \times \pi)(\cos 120\pi t)$$

$$\boxed{e = -0.1184 \cos 120\pi t \quad V}$$

Displacement current density & current

According to ampere circuit law,

$$\nabla \times \bar{H} = \bar{J}$$

taking divergence,

$$\nabla \cdot (\nabla \times \bar{H}) = \nabla \cdot \bar{J}$$

W.K.T, Divergence of the curl of any vector field is zero.

$$\nabla \cdot (\nabla \times \bar{H}) = \nabla \cdot \bar{J} = 0$$

Equation of continuity,

$$\nabla \cdot \bar{J} = -\frac{\partial \rho_v}{\partial t}$$

The above equation is not compatible for time varying field. Then

$$\nabla \times \bar{H} = \bar{J} + \bar{N}$$

taking divergence.

$$\nabla \cdot (\nabla \times \bar{H}) = \nabla \cdot \bar{J} + \nabla \cdot \bar{N} = 0$$

As $\nabla \cdot \bar{J} = -\frac{\partial \rho_v}{\partial t}$

then, $\nabla \cdot \bar{N} = +\frac{\partial \rho_v}{\partial t}$

According to gauss's law,

$$\rho_v = \nabla \cdot \bar{D}$$

apply in above equation.

$$\nabla \cdot \bar{N} = \frac{\partial}{\partial t}(\nabla \cdot \bar{D})$$

$$\nabla \cdot \bar{N} = \nabla \cdot \frac{\partial \bar{D}}{\partial t}$$

Comparing two sides of equations.

$$\bar{N} = \frac{\partial \bar{D}}{\partial t}$$

We can write ampere's circuit law in point form,

$$\nabla \times \bar{H} = J_c + \frac{\partial \bar{D}}{\partial t}$$

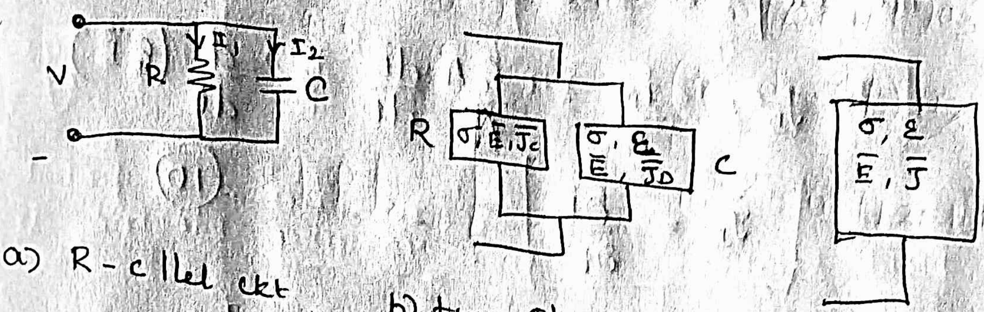
Where $J_c \rightarrow$ conduction current density.

and $J_D = \frac{\partial \bar{D}}{\partial t}$

$J_D \rightarrow$ Displacement current density

$$\nabla \times \bar{H} = \bar{J_c} + \bar{J_D}$$

Significance of Displacement current



a) R-C llel ckt b) two element c) Combined

current through Resistance $i_1 = \frac{V}{R}$ — ①

$i_1 \rightarrow$ conduction current

$i_1 = i_c$ — ②

current density $\bar{J_c} = \frac{i_c}{A} = \sigma \bar{E}$ — ③

current through capacitor $i_2 = C\frac{dV}{dt}$ — ④

W.K.T $C = \frac{\varepsilon A}{d'}$ — ⑤

$i_2 = \frac{\varepsilon A}{d'} \cdot \frac{dv}{dt}$ — ⑥

$i_2 \rightarrow$ displacement current $= i_D$

The electric field produced by the voltage applied b/w the two plates,

$E = \frac{V}{d'}$ — ⑦

$V = E.d'$ — ⑧

then $i_2 = i_D = \frac{\varepsilon A}{d'} \frac{d}{dt}(d'E)$

$i_D = \frac{\varepsilon A}{d'} d' \cdot \frac{dE}{dt}$

$$\boxed{i_D = \varepsilon A \frac{dE}{dt}}$$ — ⑨

The displacement current density

$$J_D = \frac{i_D}{A} = \frac{\varepsilon A}{A}\frac{d\bar{E}}{dt} = \frac{d}{dt}(\varepsilon\bar{E})$$

$$\boxed{\bar{J}_D = \frac{\partial \bar{D}}{\partial t}} \quad \text{——— (10)}$$

The total current density is given by

$$\boxed{\bar{J} = \bar{J}_c + \bar{J}_D}$$

$$\bar{J} = \sigma\bar{E} + \frac{\partial}{\partial t}(\varepsilon\bar{E})$$

$$\bar{J} = \sigma\bar{E} + j\omega\varepsilon\bar{E}$$ (Let time impedence $e^{j\omega t}$

$$\boxed{\frac{\bar{J}_c}{\bar{J}_D} = \frac{\sigma}{\omega\varepsilon}}$$

When $\frac{\sigma}{\omega\varepsilon} >> 1$ → medium is conductor.

$\frac{\sigma}{\omega\varepsilon} << 1$ → medium is dielectric.

In a material for which $\sigma = 5.0$ S/m and $\varepsilon_r = 1$, the electric field intensity is $E = 250 \sin 10^{10} t$ V/m. Find the conduction and displacement current densities and the frequency at which both have equal magnitudes.

conduction current density $J_c = \sigma\bar{E}$

$$J_c = 5(250 \sin 10^{10} t)$$

$$= 1250 \sin 10^{10} t \quad A/m^2$$

Displacement current density

$$J_D = \frac{\partial D}{\partial t} = \frac{\partial}{\partial t}(\varepsilon_0 E) = \frac{\partial}{\partial t}\left[\varepsilon_0 \varepsilon_r E\right]$$

$$= \frac{\partial}{\partial t}\left[8.854 \times 10^{-12} \times 1 \times 250 \times \sin 10^{10} t\right]$$

$$= \left(8.854 \times 10^{-12} \times 250\right) \times \left(10^{10}\right)\left(\cos 10^{10} t\right)$$

$$J_D = 22.135 \cos 10^{10} t \quad A/m^2$$

Frequency, ω

$$\frac{|J_c|}{|J_D|} = \frac{\sigma}{\varepsilon \omega} = 1$$

$$\omega = \frac{\sigma}{\varepsilon} = \frac{5}{8.854 \times 10^{-12} \times 1}$$

$$\omega = 5.647 \times 10^{11}$$

$$\omega = 2\pi f \Rightarrow f = \frac{\omega}{2\pi}$$

$$\boxed{f = 89.87\ GHz}$$

A parallel plate capacitor with plate area of 5 cm^2 and plate seperation of 3 mm has a voltage of $50 \sin 10^3 t$ V applied to its plates. Calculate the displacement current. assume $\varepsilon = 2\varepsilon_0$

Displacement current $I_D = \frac{\partial D}{\partial t} = \frac{\partial}{\partial t}(\varepsilon E)$

$$I_D = \frac{\partial}{\partial t}(\varepsilon_0 \varepsilon_r E) = \varepsilon_0 \varepsilon_r \frac{\partial}{\partial t}\left(\frac{V}{d}\right)$$

$$= \frac{\varepsilon_0 \varepsilon_r}{d}\frac{\partial V}{\partial t} = \frac{8.854 \times 10^{-12} \times 2}{3 \times 10^{-3}} \frac{\partial}{\partial t}\left[50 \sin 10^3 t\right]$$

$$= \frac{8.854 \times 10^{-12} \times 2 \times 50 \times 10^3 \cos 10^3 t}{3 \times 10^{-3}}$$

$$I_D = 0.2951 \times 10^{-3} \cos 10^3 t \quad A/m^2$$

$I_D = A \cdot J_D = 5 \times 10^{-4} \times 0.2951 \times 10^{-3} \cos 10^3 t$

$$\boxed{I_D = 0.1476 \times 10^{-6} \cos 10^3 t \ A}$$

Field Relations for time varying Electric and magnetic fields:-

The charge disappears from one point, then it must reappear at some other point.
This basic property is called conservation of charge.

Equation of continuity for time varying Fields:-

Faraday's law,

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

and $\bar{B} = \nabla \times \bar{A}$, $\bar{A} \rightarrow$ Vector magnetic potential

$$\nabla \times \bar{E} = -\frac{\partial}{\partial t}(\nabla \times \bar{A})$$

$$\nabla \times \bar{E} = -\nabla \times \frac{\partial \bar{A}}{\partial t}$$

$$\nabla \times \bar{E} + \nabla \times \frac{\partial \bar{A}}{\partial t} = 0$$

$$\nabla \times \left(\bar{E} + \frac{\partial \bar{A}}{\partial t}\right) = 0$$

Curl of a gradient of a scalar is always zero

$$\bar{E} + \frac{\partial \bar{A}}{\partial t} = \nabla V$$

$$\bar{E} = -\nabla V - \frac{\partial \bar{A}}{\partial t}$$

when the field is static, $\frac{\partial \bar{A}}{\partial t} = 0$

$$\bar{E} = -\nabla V$$

consider any closed surface. If the current is flowing out of the surface.

$$I = \frac{dQ}{dt} \quad A$$

Let $Q_I \rightarrow$ internal charge.

$$I = -\frac{dQ_I}{dt}$$

If there is a volume charge ρ_v,

$$Q_I = \int_V \rho_v \, dv$$

$$I = -\frac{d}{dt}\left[\int_V \rho_v \, dv\right]$$

$$I = -\int_V \frac{d\rho_v}{dt} \cdot dv$$

current can be expressed as

$$I = \int_V J \cdot d\bar{s}$$

Equate above equation.

$$-\int_V \frac{d\rho_v}{dt} \cdot dv = \int_V J \cdot d\bar{s}$$

using divergence Theorem, converting surface integral to volume integral

$$\int_V \nabla \cdot \bar{J} \, d\bar{v} = -\int_V \frac{d\rho_v}{dt} dv$$

$$\boxed{\nabla \cdot \bar{J} = -\frac{d\rho_v}{dt}}$$

This equation is called equation of continuity of current in point or differential form.

Inconsistancy of Ampere's Circuital law -

Modification in equation of Continuity:-

Consider ampere's circuit law in point or differential form,

$$\nabla \times \bar{H} = \bar{J} \quad \text{———} ①$$

taking divergence on both sides,

$$\nabla \cdot (\nabla \times \bar{H}) = \nabla \cdot \bar{J} \quad \text{———} ②$$

According to vector velocity, divergence of curl of Vector is zero,

$$\therefore \nabla \cdot \bar{J} = 0 \quad (\text{Static fields}) \quad \text{———} ③$$

This result is not consistent with the continuity equation.

$$\nabla \cdot \bar{J} = \frac{d \rho_v}{dt} \quad \text{———} ④$$

In other words, Ampere's circuit law is not consistent and need some modification.

Ampere's circuit law for time varying fields

$$\nabla \times \bar{H} = \bar{J} + \bar{U} \quad \text{———} ⑤$$

Taking divergence on both sides,

$$\nabla \cdot (\nabla \times \bar{H}) = \nabla \cdot (\bar{J} + \bar{U})$$

But $\nabla \cdot (\nabla \times \bar{H}) = 0$

then $\nabla \cdot (\bar{J} + \bar{U}) = 0$

$$\nabla \cdot \bar{J} + \nabla \cdot \bar{U} = 0$$

$$\nabla \cdot \bar{J} = - \nabla \cdot \bar{U} \quad \text{———} ⑥$$

From the continuity equation.

$$\nabla \cdot \bar{J} = -\frac{d \rho_v}{dt}$$

$$\nabla \cdot \bar{U} = -\left(-\frac{d \rho_v}{dt}\right)$$

$$\boxed{\nabla \cdot \bar{U} = \frac{d \rho_v}{dt}} \quad \text{--- (7)}$$

From gauss's law in point form,

$$\nabla \cdot \bar{D} = \rho_v$$

$$\nabla \cdot \frac{d \bar{D}}{dt} = \frac{d \rho_v}{dt}$$

$$\nabla \cdot \frac{d \bar{D}}{dt} = \nabla \cdot \bar{U}$$

$$\boxed{\bar{U} = \frac{d \bar{D}}{dt}} \quad \text{--- (8)}$$

time varying field, ampere's circuit law can be written as,

$$\boxed{\nabla \times \bar{H} = \bar{J} + \frac{d \bar{D}}{dt}}$$

Maxwell's Equations

Maxwell's equations are nothing but a set of four expressions

- Ampere's circuit law
- Faraday's law
- Gauss's law for electric field
- Gauss's law for magnetic field

Maxwell's equation in integral form

It is govern the independence of fields like $\bar{E}$, $\bar{D}$, $\bar{B}$ and $\bar{H}$, along with sources of fields like charge and current associated with different regions in the space like surface and volumes.

Maxwell's equation in differential form:

It explain the characteristics of different field vectors at a given point to each other as well as to the charge and current densities at that point.

Maxwell's Equations for static Fields

a) Maxwell's equation derived from faraday's law

According to the basic concepts from an electro static field, the workdone over a closed path (or) closed contour is always zero.

$$\oint \bar{E} \cdot d\bar{L} = 0$$

This is called integral form of maxwell's equation derived from Faraday's law for static field.

stoke's theorem converting the closed line integral into the surface integral, we get.

$$\oint \bar{E}\cdot d\bar{L} = \int_S (\nabla \times \bar{E})\cdot d\bar{s} = 0$$

$$\int_S (\nabla \times \bar{E})\cdot d\bar{s} = 0$$

But $d\bar{s}$ cannot be zero

$$\nabla \times \bar{E} = 0$$

This is called point or differential form of maxwell's equation derived from Faraday's law for static fields.

b) Maxwell's equation Derived from Ampere's Circuit law:

According to basic concepts from magnetostatics an ampere's circuit law states that

the line integral of magnetic field intensity $\bar{H}$ around a closed path is exactly equal to the direct current enclosed by the path

$$\oint \bar{H}\cdot d\bar{L} = I \quad \text{---} (1)$$

The current enclosed is equal to the product of current density normal to closed path and area of closed path.

$$I = \int_S \bar{J}\cdot d\bar{s} \quad \text{---} (2).$$

Equate eqn (1) & (2), we get

$$\boxed{\oint \bar{H}\cdot d\bar{L} = \int_S \bar{J}\cdot d\bar{s}}$$

This is integral form of maxwell's equation derived from ampere's circuital law for static field.

Stokes' theorem,

$$\oint \bar{H}\cdot d\bar{L} = \int_S (\nabla\times\bar{H})\cdot d\bar{s} = \int_S \bar{J}\cdot d\bar{s}$$

$$\therefore \boxed{\nabla\times\bar{H} = \bar{J}}$$

This is called point or differential form of maxwell's equation derived from Ampere's circuit law for static field.

c) Maxwell's equation Derived from Gauss's law for Electrostatic fields:

According to Gauss's law for electrostatic field, the electric flux passing through any closed surface is equal to the total charge enclosed by that surface.

$$\psi = \oint \bar{D}\cdot d\bar{s} = Q_{enclosed} \quad \text{—— (1)}$$

Gauss's law with volume charge density,

$$\oint \bar{D}\cdot d\bar{s} = \int_V \rho_v \, dv \quad \text{—— (2)}$$

This is integral form of maxwell's equation derived from Gauss's law for static electric field.

The relationship between $\bar{D}$ and ρ_v, converting closed surface integral into volume integral using divergence Theorem,

$$\oint \bar{D} \cdot d\bar{s} = \int_V (\nabla \cdot \bar{D}) \, dv \quad \text{---} (3)$$

Compare equation (2) & (3) we get

$$\int_V (\nabla \cdot \bar{D}) \, dv = \int_V \rho_v \, . dv$$

$$\boxed{\nabla \cdot \bar{D} = \rho_v}$$

This is called point or differential form of maxwell's equation derived from Gauss's law for static electric field.

d) Maxwell's Equation Derived from Gauss's law For magnetostatic field:

According to the Gauss's law for the magnetostatic field, the magnetic flux cannot reside in a closed surface due to the non existance of single magnetic pole.

$$\oint_S \bar{B} \cdot d\bar{s} = 0$$

This is called integral form of maxwell's equation derived from gauss's law for static magnetic field.

Using divergence theorem,

$$\oint_S \bar{B} \cdot d\bar{s} = \int_V (\nabla \cdot \bar{B}) \, . dv = 0$$

$$\int_V (\nabla \cdot \bar{B}) \, dv = 0$$

Now dv cannot be zero, ie.

$$\nabla \cdot \bar{B} = 0$$

This is called point or differential form of maxwell's equation derived from gauss's law for static magnetic field.

Maxwell's Equations for Time Varying Fields

a) Maxwell's equation derived from Faraday's law

Faraday's law,

emf induced in a circuit to the time rate of decrease of total magnetic flux linking the circuit.

$$\oint \bar{E} \cdot d\bar{L} = -\int_S \frac{\partial \bar{B}}{\partial t} \cdot d\bar{s}$$

This is maxwell's equation derived from Faraday's law expressed in integral form.

Stoke's Theorem:-

"The total electromotive force (emf) induced in a closed path is equal to the negative surface integral of the rate of change of flux density with respect to time over an entire surface bounded by the same closed path"

Using stokes theorem,

$$\int_S (\nabla \times \bar{E}) \cdot d\bar{s} = -\int_S \frac{\partial \bar{B}}{\partial t} \cdot d\bar{s}$$

Assume that the integration is carried out over the same surface on both the sides.

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

This is maxwell's equation derived from Faraday's law expressed in point form or differential form.

b) <u>Maxwell's Equation Derived from Ampere's Circuit law</u>

According to Ampere's circuit law, the line integral of magnetic field intensity $\bar{H}$ around a closed path is equal to the current enclosed by the path.

$$\oint \bar{H} \cdot d\bar{l} = I_{enclosed}.$$

Replacing current by the surface integral of conduction current density $\bar{J}$ over an area bounded by the path of integration of $\bar{H}$,

$$\oint \bar{H} \cdot d\bar{l} = \int_S \bar{J} \cdot d\bar{s}$$

By adding displacement current density to conduction current density,

$$\oint \bar{H} \cdot d\bar{l} = \int_S \left[\bar{J} + \frac{\partial \bar{D}}{\partial t} \right] \cdot d\bar{s}$$

This is maxwell's equations derived from Ampere's circuit law. This integral form.

Statement:

The total magnetomotive force around any closed path is equal to the surface integral of the conduction and displacement current densities over the entire surface bounded by the same closed path.

Apply stokes theorem

$$\int_S (\nabla \times \bar{H}) \cdot d\bar{s} = \int_S \left(\bar{J} \cdot \frac{\partial \bar{D}}{\partial t} \right) \cdot d\bar{s}$$

Assuming that the surface considered for both the integration is same, we can write

$$\boxed{\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}}$$

This is point form or differential form of maxwell's equation derived from Ampere's Circuit law.

(c) <u>Maxwell's equation</u> <u>Derived from Gauss's law for Electric Field</u>:-

According to gauss's law, the total flux out of the closed path surface is equal to the net charge with in the surface.

$$\int_S \bar{D} \cdot d\bar{s} = Q_{enclosed}$$

The volume integral of volume charge density ρ_v through the volume enclosed by the surface S considered for integration

$$\int_S \bar{D} \cdot d\bar{s} = \int_V \rho_v \, dv.$$

This is called Maxwell's equation for electric fields derived from Gauss's law in integral form.

statement:- The total flux leaving out of a closed surface is equal to the total charge enclosed by a finite volume.

using divergence theorem,

$$\int_V (\nabla \cdot \bar{D}) \, dv = \int_V \rho_v \, dv$$

Assume same volume for integration on both sides

$$\nabla \cdot \bar{D} = \rho_v .$$

This is maxwell's equation for electric fields derived from gauss's law which is expressed in point form or differential form.

d) <u>Maxwell's</u> <u>equation</u> <u>Derived</u> <u>from</u> <u>gauss's</u> <u>law</u> <u>magnetic</u> <u>fields</u>:-

For magnetic fields, the surface integral of $\bar{B}$ over a closed surface s is always zero, due to non existance of monopole in the magnetic fields.

$$\int_s \bar{B} \cdot d\bar{s} = 0 .$$

This is the integral form of maxwell's magnetic field equation expressed.

statement: The surface integral of magnetic flux density over a closed surface is always equal to zero,

Using divergence theorem,

$$\int_s (\nabla \cdot \bar{B}) \, dv = 0$$

Being a finite volume, $dv \neq 0$

$$\nabla \cdot \bar{B} = 0$$

This is differential form or point form of Maxwell's equation derived from Gauss's law applied to the magnetic fields.

Maxwell's Equations for free space:

Point form	Integral Form
$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$	$\oint \bar{E} \cdot d\bar{l} = -\int_S \frac{\partial \bar{B}}{\partial t} \cdot d\bar{s}$
$\nabla \times \bar{H} = +\frac{\partial \bar{D}}{\partial t}$	$\oint \bar{H} \cdot d\bar{l} = \int_S \frac{\partial \bar{D}}{\partial t} \cdot d\bar{s}$
$\nabla \cdot \bar{D} = 0$	$\oint \bar{D} \cdot d\bar{s} = 0$
$\nabla \cdot \bar{B} = 0$	$\oint \bar{B} \cdot d\bar{s} = 0$

Maxwell's equations for Good conductor

Point Form	Integral Form
$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$	$\oint \bar{E} \cdot d\bar{l} = -\int_S \frac{\partial \bar{B}}{\partial t} \cdot d\bar{s}$
$\nabla \times \bar{H} = \bar{J}$	$\oint \bar{H} \cdot d\bar{l} = I = \int_S \bar{J} \cdot d\bar{s}$
$\nabla \cdot \bar{D} = 0$	$\oint_S \bar{D} \cdot d\bar{s} = 0$
$\nabla \cdot \bar{B} = 0$	$\oint \bar{B} \cdot d\bar{s} = 0.$

Maxwell's equations for Harmonically Varying Fields:-

Point Form:-

1) $\nabla \times \bar{E} = -j\omega \bar{B} = -j\omega\mu \bar{H}$

2) $\nabla \times \bar{H} = \bar{J} + j\omega \bar{D} = \sigma \bar{E} + j\omega(\epsilon \bar{E})$

3) $\nabla \cdot \bar{D} = \rho_v$

4) $\nabla \cdot \bar{B} = 0$

B) Integral Form

1) $\oint \bar{E} \cdot d\bar{L} = -\int_S j\omega \bar{B} \cdot d\bar{s} = -\int j\omega \mu \bar{H} \cdot ds$

2) $\oint \bar{H} \cdot d\bar{L} = I + \int_S j\omega \bar{D} \cdot d\bar{s} = (\sigma + j\omega \epsilon) \int_S \bar{E} \cdot d\bar{s}$

3) $\oint \bar{D} \cdot d\bar{s} = \oint_V \rho_v \, dv$

4) $\oint \bar{B} \cdot d\bar{s} = 0$

Comparison Between Field Theory & Circuit Theory

$$\vec{E}_{total} = \bar{E}_e + \bar{E} \quad \text{——— (1)}$$

$\bar{E}_e \rightarrow$ electric field related to emf

$\vec{E} \rightarrow$ Electric field due to charges & current.

$$\bar{E}_e = \vec{E}_{total} - \bar{E} \quad \text{——— (2)}$$

and

$$\bar{E}_{total} = \frac{\bar{J}}{\sigma} \quad \text{——— (3)}$$

then

$$\bar{E} = -\nabla V - \frac{\partial \bar{A}}{\partial t} \quad \text{——— (4)}$$

Sub eqn (3) & eqn (4) in eqn (2).

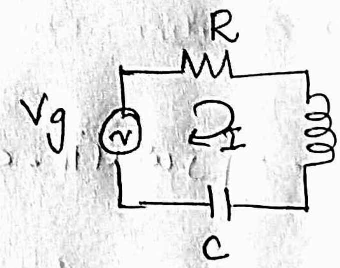
$$\bar{E}_e = \frac{\bar{J}}{\sigma} - \left[-\nabla V - \frac{\partial \bar{A}}{\partial t} \right]$$

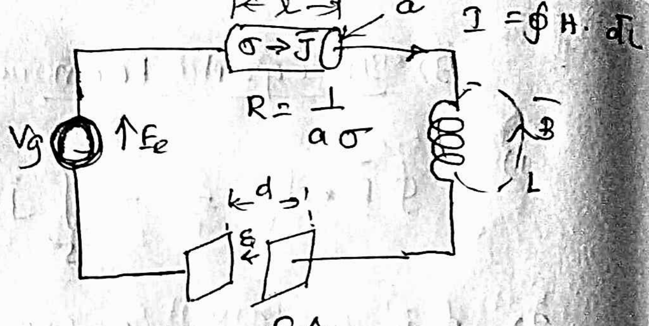
$$\bar{E}_e = \frac{\bar{J}}{\sigma} + \nabla V + \frac{\partial \bar{A}}{\partial t} \quad \text{——— (5)}$$

Integrating the above eqn,

$$\oint \bar{E}_e \cdot d\bar{L} = \oint \frac{\bar{J}}{\sigma} \cdot d\bar{L} + \oint \nabla V \cdot d\bar{L} + \oint \frac{\partial \bar{A}}{\partial t} \cdot d\bar{L}$$

$$V_g = \frac{\bar{J}}{\sigma} \cdot l + E_d + \frac{d}{dt} \oint \bar{A} \cdot dL \quad \text{——— (6)}$$





Now consider last form,

$$\frac{d}{dt}\oint \bar{A}\cdot d\bar{l} = \frac{d}{dt}\int_S (\nabla\times\bar{A})\cdot d\bar{s}$$

$$= \frac{d}{dt}\int_S \bar{B}\cdot d\bar{s} = \frac{d\varphi}{dt} = L\cdot\frac{dI}{dt}$$

Hence,

$$V_g = \frac{I}{a}\left(\frac{l}{\sigma}\right) + \frac{Dd}{\varepsilon} + L\frac{dI}{dt} \qquad \because J = \frac{I}{a}$$

$$E = \frac{D}{\varepsilon}$$

Now $\frac{l}{a\sigma} = R$, $D = \frac{Q}{A}$

hence

$$V_g = IR + \frac{Q}{A\varepsilon/d} + L\cdot\frac{dI}{dt}$$

$\because$ $C = \frac{A\varepsilon}{d}$ & $Q = \int I\cdot dt$, becomes

$$\boxed{V_g = IR + \frac{1}{C}\int I\,dt + L\frac{dI}{dt}}$$

x x

Unit - 5

Electromagnetic Waves

The waves are the means of transporting energy or information from source to destination. The waves are consisting electric and magnetic fields are called electromagnetic waves.

The wave is a function of time and space. Ex. Radio waves, light rays, radar beams, television signals etc.

Properties:

* They assume properties of waves while travelling.
* They travel with high velocity.
* They radiate outwards from source in all directions.

Travelling media:

* ~~Lossloss~~ media
  * Perfect dielectries
  * Free space
* Lossy media
  * Good dielectric
  * Good conductor,
  * ~~Good~~ Perfect conductor.

Single continuous medium

* Lossless medium
  * Perfect dielectrics: $\sigma = 0$, $\mu = \mu_0 \cdot \mu_r$, $\varepsilon = \varepsilon_0 \varepsilon_r$
  * free space: $\sigma = 0$, $\mu = \mu_0$, $\varepsilon = \varepsilon_0 \varepsilon_r$
* lossy medium
  * Good dielectric: $\sigma \neq 0$, $\mu = \mu_0 \mu_r$, $\varepsilon = \varepsilon_0 \times \varepsilon_r$
  * Good conductor: $\sigma \neq 0$, $\mu = \mu_0 \mu_r$, $\varepsilon = \varepsilon_0 \times$
  * Perfect conductor or $\sigma = \alpha$, $\mu = \mu_0 \mu$, $\varepsilon = \varepsilon_0$

General wave Equation:-

Let us assume that, the electric and magnetic fields exists in a linear, homogeneous and isotropic medium with the parameters μ, ϵ & σ

Assume charge free medium, it obeys ohm's law

ie $\bar{J} = \sigma \bar{E}$

Then maxwell's equations are given by

$$\nabla \times \bar{E} = -\mu \frac{\partial \bar{H}}{\partial t} \quad \text{——— (1)}$$

$$\nabla \times \bar{H} = \sigma \bar{E} + \epsilon \frac{\partial \bar{E}}{\partial t} \quad \text{——— (2)}$$

$$\nabla \cdot \bar{B} = 0, \quad \text{ie } \nabla \cdot \bar{H} = 0 \quad \text{——— (3)}$$

$$\nabla \cdot \bar{D} = 0 \quad \text{ie } \nabla \cdot \bar{E} = 0 \quad \text{——— (4)}$$

From eqn (1),

$$\nabla \times \nabla \times \bar{E} = -\mu \left[\nabla \times \frac{\partial \bar{H}}{\partial t} \right] \quad \text{——— (5)}$$

Interchange ∇ & $\frac{\partial}{\partial t}$ in R.H.S

$$\nabla \times \nabla \times \bar{E} = -\mu \left[\frac{\partial}{\partial t} (\nabla \times \bar{H}) \right] \quad \text{——— (6)}$$

Sub eqn (2) in eqn (6)

$$\nabla \times \nabla \times \bar{E} = -\mu \frac{\partial}{\partial t} \left[\sigma \bar{E} + \epsilon \frac{\partial \bar{E}}{\partial t} \right]$$

$$\nabla \times \nabla \times \bar{E} = -\mu \sigma \frac{\partial \bar{E}}{\partial t} - \mu \epsilon \frac{\partial^2 \bar{E}}{\partial t^2} \quad \text{——— (7)}$$

According to vector identity,

$$\nabla \times \nabla \times \bar{E} = \nabla (\nabla \cdot \bar{E}) - \nabla^2 \bar{E} \quad \text{——— (8)}$$

Sub, $\nabla \cdot \bar{E} = 0$ in eqn (8)

$$\nabla \times \nabla \times \bar{E} = -\nabla^2 \bar{E} \quad \text{——— (9)}$$

Sub eqn ⑨ in eqn ⑦

$$-\nabla^2 E = -\mu\sigma \frac{\partial \bar{E}}{\partial t} - \mu\varepsilon \frac{\partial^2 \bar{E}}{\partial t^2}$$

$$\boxed{\nabla^2 E = \mu\sigma \frac{\partial \bar{E}}{\partial t} + \mu\varepsilon \frac{\partial^2 \bar{E}}{\partial t^2}} \quad \text{——— ⑩}$$

This is the wave equation for the electric field $\bar{E}$.

The above equation multiply by 'ε'

$$\nabla^2 (\varepsilon E) = \mu\sigma \frac{\partial \varepsilon \bar{E}}{\partial t} + \mu\varepsilon \frac{\partial^2 (\varepsilon \bar{E})}{\partial t^2}$$

$$\therefore \boxed{\nabla^2 \bar{D} = \mu\sigma \frac{\partial \bar{D}}{\partial t} + \mu\varepsilon \frac{\partial^2 \bar{D}}{\partial t^2}} \quad \text{——— ⑪}$$

This the wave equation of $\bar{D}$ in uniform medium.

From eqn ②,

$$\nabla \times (\nabla \times \bar{H}) = \nabla \times (\sigma \bar{E}) + \varepsilon \nabla \times \frac{\partial \bar{E}}{\partial t} \quad \text{——— ⑫}$$

Interchange ∇ & $\frac{\partial}{\partial t}$

$$\nabla \times (\nabla \times \bar{H}) = \nabla \times (\sigma \bar{E}) + \varepsilon \cdot \frac{\partial}{\partial t} (\nabla \times \bar{E}) \quad \text{— ⑬}$$

Sub $\nabla \times E = -\mu \frac{\partial \bar{H}}{\partial t}$

$$\nabla \times (\nabla \times \bar{H}) = \sigma \left[-\mu \cdot \frac{\partial \bar{H}}{\partial t} \right] + \varepsilon \frac{\partial}{\partial t} \left[-\mu \frac{\partial \bar{H}}{\partial t} \right]$$

$$\nabla \times \nabla \times \bar{H} = -\mu\sigma \frac{\partial \bar{H}}{\partial t} - \mu\varepsilon \frac{\partial^2 \bar{H}}{\partial t^2} \quad \text{——— ⑭}$$

From the vector identity,

$$\nabla \times \nabla \times \bar{H} = \nabla (\nabla \cdot \bar{H}) - \nabla^2 \bar{H} \quad \text{——— ⑮}$$

Sub $\nabla \times \bar{H} = 0$, then

$$\nabla \times \nabla \times \bar{H} = -\nabla^2 \bar{H} \quad \text{——— ⑯}$$

Equate eqn ⑯ & 14

$$-\nabla^2 \bar{H} = -\mu\sigma \frac{\partial \bar{H}}{\partial t} - \mu\varepsilon \frac{\partial^2 \bar{H}}{\partial t^2}$$

$$\boxed{\nabla^2 \bar{H} = \mu\sigma \frac{\partial \bar{H}}{\partial t} + \mu\varepsilon \frac{\partial^2 \bar{H}}{\partial t^2}} \quad \text{--- (17)}$$

This is wave equation for the magnetic field $\bar{H}$.

Multiply both sides by μ

$$\nabla^2 (\mu \bar{H}) = \mu\sigma \frac{\partial (\mu \bar{H})}{\partial t} + \mu\varepsilon \frac{\partial^2 (\mu \bar{H})}{\partial t^2}$$

$$\boxed{\nabla^2 \bar{B} = \mu\sigma \frac{\partial \bar{B}}{\partial t} + \mu\varepsilon \frac{\partial^2 \bar{B}}{\partial t^2}} \quad \text{--- (18)}$$

This is the wave equation for $\bar{B}$ in the uniform medium.

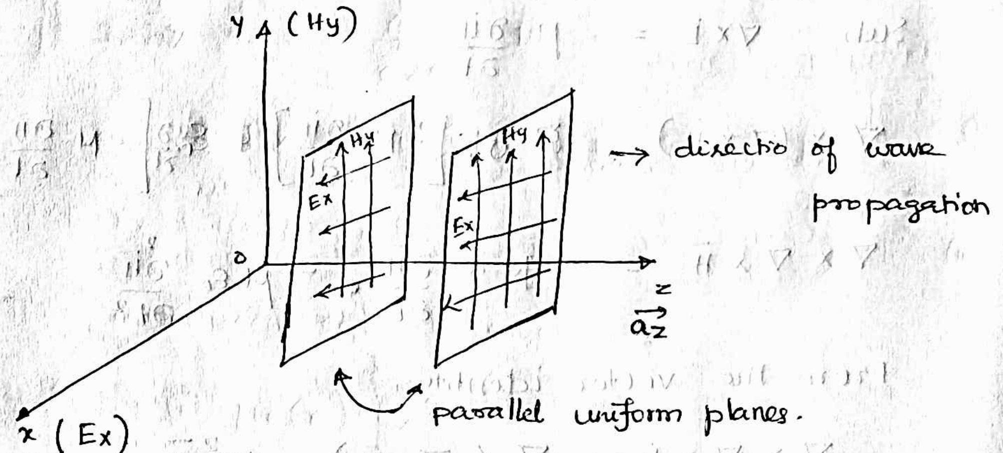
In general,

$$\nabla^2 \begin{bmatrix} \bar{E} \\ \bar{D} \\ \bar{H} \\ \bar{B} \end{bmatrix} = \mu\sigma \frac{\partial}{\partial t} \begin{bmatrix} \bar{E} \\ \bar{D} \\ \bar{H} \\ \bar{B} \end{bmatrix} + \mu\varepsilon \frac{\partial^2}{\partial t^2} \begin{bmatrix} \bar{E} \\ \bar{D} \\ \bar{H} \\ \bar{B} \end{bmatrix}$$

x x x

Uniform Plane waves in Free Space :-

For free space $\sigma = 0$



The electromagnetic waves are also called as TransVerse electro magnetic waves.

$\bar{E}$ & $\bar{H}$ is perpendicular to each other.

Let us consider wave equations $\bar{E}$ & $\bar{H}$ is given by

$$\nabla^2 \bar{E} = \mu\sigma \frac{\partial \bar{E}}{\partial t} + \mu\varepsilon \frac{\partial^2 \bar{E}}{\partial t^2} \quad \text{--- (1)}$$

$$\nabla^2 \bar{H} = \mu \sigma \frac{\partial \bar{H}}{\partial t} + \mu \varepsilon \frac{\partial^2 \bar{H}}{\partial t^2} \quad \text{——— (2)}$$

For free space, $\sigma = 0$, $\varepsilon = \varepsilon_0$, $\mu = \mu_0$, then

$$\nabla^2 \bar{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \bar{E}}{\partial t^2} \quad \text{——— (3)}$$

$$\nabla^2 H = \mu_0 \varepsilon_0 \frac{\partial^2 \bar{H}}{\partial t^2} \quad \text{——— (4)}$$

from eqn (3),

$$\nabla^2 \bar{E} = \frac{\partial^2 \bar{E}}{\partial x^2} + \frac{\partial^2 \bar{E}}{\partial y^2} + \frac{\partial^2 \bar{E}}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 \bar{E}}{\partial t^2} \quad \text{——— (5)}$$

The wave travels in z - direction.

$$\frac{\partial^2 \bar{E}}{\partial z^2} = \mu_0 \varepsilon_0 \cdot \frac{\partial^2 \bar{E}}{\partial t^2}$$

$$\boxed{\frac{\partial^2 \bar{E}}{\partial t^2} = \frac{1}{\mu_0 \varepsilon_0} \frac{\partial^2 \bar{E}}{\partial t^2}} \quad \text{——— (6)}$$

From basics of physics,

$$v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = c, \quad v^2 = \frac{1}{\mu \varepsilon} = c^2$$

$c \rightarrow 3 \times 10^8$ m/s $\rightarrow$ velocity of light.

Sub. in eqn (6),

$$\frac{\partial^2 \bar{E}}{\partial t^2} = v^2 \cdot \frac{\partial^2 E}{\partial z^2} \quad \text{——— (7)}$$

The above equation is the other form of wave equation,

$$\frac{\partial^2 \bar{H}}{\partial t^2} = v^2 \frac{\partial^2 \bar{H}}{\partial z^2} \quad \text{——— (8)}$$

Let us consider eqn

$$\frac{\partial^2 \bar{E}}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 \bar{E}}{\partial t^2} \quad \text{——— (9)}$$

According to assumption, $\bar{E}$ is direction.

$$\frac{\partial^2 E_x}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_x}{\partial t^2} \quad \text{—— (10)}$$

$$\text{Let } E_x = E_m e^{j\omega t} \quad \text{—— (11)}$$

$E_m \rightarrow$ amplitude of Electric field

$\omega \rightarrow$ angular frequency

$$\frac{\partial^2 E_x}{\partial t^2} = E_m (j\omega)(j\omega) e^{j\omega t}$$

$$= -\omega^2 E_m e^{j\omega t} \quad \text{—— (12)}$$

But $E_m e^{j\omega t} = E_x$.

$$\frac{\partial^2 E_x}{\partial t^2} = -\omega^2 E_x \quad \text{—— (13)}$$

apply in eqn (10).

$$\frac{\partial^2 E_x}{\partial z^2} = \mu_0 \varepsilon_0 (-\omega^2 E_x)$$

$$= -\omega^2 \mu_0 \varepsilon_0 E_x \quad \text{—— (14)}$$

Let $\frac{\partial}{\partial z} = D$, $\frac{\partial^2}{\partial z^2} = D^2$

$$D^2 E_x = -\omega^2 \mu_0 \varepsilon_0 E_x \quad \text{—— (15)}$$

$$D^2 E_x + \omega^2 \mu_0 \varepsilon_0 E_x = 0 \quad \text{—— (16)}$$

Thus auxiliary eqn becomes,

$$(D^2 + \omega^2 \mu_0 \varepsilon_0) E_x = 0.$$

$$D^2 = -\omega^2 \mu_0 \varepsilon_0$$

$$D = \pm j\omega \sqrt{\mu_0 \varepsilon_0} = \pm j\beta$$

$$\text{where } \beta = \omega \sqrt{\mu_0 \varepsilon_0} \quad \text{—— (17)}$$

which is called phase shift constant measured in rad/sec.

Hence the solution of equation (12),

$$E_x = k_1 e^{-j\omega\sqrt{\mu_0 \varepsilon_0}\, z} + k_2 e^{+j\omega\sqrt{\mu_0 \varepsilon_0}\, z}$$

$$E_x = k_1 e^{-j\beta z} + k_2 e^{j\beta z} \quad \text{——— (18)}$$

Let k_1 & k_2 be constant. Let assume,

$$k_1 = E_m^+ e^{+j\omega t}, \quad k_2 = E_m^- e^{+j\omega t}$$

$$\therefore E_x = E_m^+ e^{+j\omega t} e^{-j\beta z} + E_m^- e^{j\omega t} \cdot e^{j\beta z}$$

$$E_x = E_m^+ e^{j(\omega t - \beta z)} + E_m^- e^{j(\omega t + \beta z)}$$

$$\therefore \boxed{E_x = E_m^+ \cos(\omega t - \beta z) + E_m^- \cos(\omega t + \beta z)} \ V/m. \quad \text{——— (19)}$$

Similarly

$$H_y = H_y^+ \cos(\omega t - \beta z) + H_y^- \cos(\omega t + \beta z) \quad A/m. \quad \text{——— (20)}$$

Phase velocity:

The phase velocity of the uniform plane waves is defined as the velocity with which the place of wave propagates. (v_p).

$$v_p = \frac{1}{\sqrt{\mu_0 \times \varepsilon_0}} = 3 \times 10^8 \ m/s = c$$

Group velocity:-

The velocity of entire group of waves as a whole is called group velocity (v_g)

$$v_g = v_p - \lambda \frac{dv_p}{d\lambda}$$

when $\frac{dv_p}{d\lambda} = 0$, then $v_g = v_p$ is called non despersive medium.

$v_g < v_p$ is called normal despersive medium.

$v_g > v_p$ is called anomolous despersive medium.

Relationship between $\bar{E}$ and $\bar{H}$ in free space.

Consider Maxwell's equation derived from Faraday's la

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} = -\mu \frac{\partial \bar{H}}{\partial t} \quad \text{--- (1)}$$

$$\begin{vmatrix} \bar{a_x} & \bar{a_y} & \bar{a_z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x & E_y & E_z \end{vmatrix} = -\mu \frac{\partial}{\partial t}\left[H_x \bar{a_x} + H_y \bar{a_y} + H_z \bar{a_z} \right] \quad \text{--- (2)}$$

Assume that uniform plane wave is propagating in z-direction.

$$\begin{bmatrix} \bar{a_x} & \bar{a_y} & \bar{a_z} \\ 0 & 0 & \partial/\partial z \\ E_x & E_y & 0 \end{bmatrix} = -\mu \frac{\partial}{\partial t}\left[H_x \bar{a_x} + H_y \bar{a_y} \right] \quad \text{--- (3)}$$

$\bar{E}$ & $\bar{H}$ is perpendicular to each other, then $E_y = 0$, $H_x =$

$$\begin{vmatrix} \bar{a_x} & \bar{a_y} & \bar{a_z} \\ 0 & 0 & \partial/\partial z \\ E_x & 0 & 0 \end{vmatrix} = -\mu \frac{\partial}{\partial t}\left[H_y \bar{a_y} \right] \quad \text{--- (4)}$$

$$\frac{\partial E_x}{\partial z} \bar{a_y} = -\mu \frac{\partial}{\partial t} \times H_y \bar{a_y}$$

$$\boxed{\frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t}} \quad \text{--- (5)}$$

Differentiating with respect to z

$$E_x = E_m^+ \cos(\omega t - \beta z) + E_m^- (\cos(\omega t + \beta z))$$

$$\frac{\partial E_x}{\partial z} = \beta E_m^+ \sin(\omega t - \beta z) - E_m^- \sin(\omega t + \beta z) \quad \text{--- (6)}$$

sub eqn (6) in eqn (5)

$$\beta E_m^+ \sin(\omega t - \beta z) - E_m^- \sin(\omega t + \beta z) = -\mu \frac{\partial H_y}{\partial t}$$

$$\frac{\partial H_y}{\partial t} = -\frac{\beta}{\mu} E_m^+ \sin(\omega t - \beta z) - E_m^- \sin(\omega t + \beta z)$$

Integrating ;

$$H_y = \frac{-\beta}{\mu}\left[-\frac{E_m^+ \cos(\omega t-\beta z)}{\omega} - \left\{\frac{E_m^- \cos(\omega t-\beta z)}{\omega}\right\}\right]$$

$$\boxed{H_y = \frac{\beta}{\mu\omega} E_m^+ \cos(\omega t-\beta z) - \frac{\beta}{\omega\mu}\cos(\omega t+\beta z).}$$ — (7)

W.K.T

$$H_y = H_m^+ \cos(\omega t-\beta z) - H_m^- \cos(\omega t+\beta z) \quad A/m.$$

Compare above 2 eqn,

$$H_m^+ = \frac{\beta}{\mu\omega} E_m^+ \qquad H_m^- = -\frac{\beta}{\mu\omega} E_m^-$$

Hence $$H_m^+ = \frac{E_m^+}{\frac{\mu\omega}{\beta}} = \frac{E_m^+}{\frac{\mu\omega}{\omega\sqrt{\mu\varepsilon_0}}} = \frac{E_m^+}{\sqrt{\frac{\mu}{\varepsilon_0}}}$$

Similarly $$H_m^- = \frac{E_m^-}{\sqrt{\mu/\varepsilon_0}}$$

The impedence is expressed interms of μ and ε_0 which are properties of the medium. Hence the impedence is called Intrensic Impedence of the medium.

$$\eta = \frac{E_m^+}{H_m^+} = -\frac{E_m^-}{H_m^-} = \sqrt{\frac{\mu}{\varepsilon_0}} \ \Omega$$

For free space $\eta_0 = 377\,\Omega$

<u>Propagation constant</u>:-

consider maxwell equation derived from faraday's law,

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} = -\mu\frac{\partial \bar{H}}{\partial t}$$

Taking curl on both sides

$$\nabla \times \nabla \times \bar{E} = -\mu\left[\nabla \times \frac{\partial \bar{H}}{\partial t}\right]$$

The above eqn simplified, and we can write

$$\nabla^2 \bar{E} = \mu \sigma \frac{\partial \bar{E}}{\partial t} + \mu \varepsilon_e \frac{\partial^2 \bar{E}}{\partial t^2}$$

By properties of phasors.

$$\nabla^2 \bar{E} = \mu \sigma . j\omega \bar{E} + \mu \varepsilon_e (j\omega)^2 \bar{E}$$

$$\boxed{\begin{array}{l} \nabla^2 \bar{E} = \left\{ j\omega\mu \left[\sigma + j\omega \varepsilon_e \right] \right\} \bar{E} \\ \text{Similarly} \\ \nabla^2 \bar{H} = \left[j\omega\mu (\sigma + j\omega \varepsilon_e) \right] \bar{H} \end{array}}$$

This is called wave equations in phasor form.

Then

$$\nabla^2 \bar{E} = \gamma^2 \bar{E}$$

$$\nabla^2 \bar{H} = \gamma^2 \bar{H}$$

where γ – propagation constant $= \sqrt{j\omega\mu(\sigma + j\omega \varepsilon_e)}$

$$\gamma = \alpha + j\beta = \sqrt{j\omega\mu(\sigma + j\omega c)}$$

$\alpha \to$ attenuation constant. (Np/m) or dB

1 Np = 8.686 dB

1 dB = 0.115 Np.

$\beta \to$ phase constant (rad/m).

$$\alpha = \omega \sqrt{\frac{\mu \varepsilon_e}{2} \left(\sqrt{1 + \left(\frac{\sigma}{\omega c}\right)^2} - 1 \right)} \ \text{Np/m}$$

$$\beta = \omega \sqrt{\frac{\mu \varepsilon_e}{2} \left[\sqrt{1 + \left(\frac{\sigma}{\omega c}\right)^2} + 1 \right]} \ \text{rad/m}$$

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega \varepsilon_e}} \ \Omega.$$

For free space, $\sigma = 0$, $\varepsilon_e = \varepsilon_0$, $\mu = \mu_0$, then.

$$\gamma = \alpha + j\beta = 0 + j\omega \sqrt{\mu_0 \varepsilon_0}$$

Then $\alpha = 0$, $\beta = \omega \sqrt{\mu_0 \varepsilon_0}$.

For free space, Propagation constant purely imaginary.

Wave Length (λ)

The distance that must be travelled by the wave to change phase by 2π radian is called wavelength (λ)

$$\lambda = \frac{2\pi}{\beta} \quad , \quad \beta = \omega\sqrt{\mu\varepsilon_0}$$

$$\lambda = \frac{\frac{1}{\sqrt{\mu\varepsilon_0}}}{\omega/2\pi} = \frac{V}{f}$$

$$\boxed{\lambda = V/f}$$

Electromagnetic wave equations in phasor form:-

consider maxwell equation derived from faraday's law

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} = -\mu \frac{\partial \bar{H}}{\partial t} \quad \text{——— ①}$$

taking curl on both sides

$$\nabla \times \nabla \times \bar{E} = -\mu\left[\nabla \times \frac{\partial \bar{H}}{\partial t}\right]$$

$$= -\mu \frac{\partial}{\partial t}(\nabla \times \bar{H}) \quad \text{——— ②}$$

Using vector identity

$$\nabla(\nabla \cdot E) - \nabla^2 E = -\mu \frac{\partial}{\partial t}(\nabla \times \bar{H}) \quad \text{——— ③}$$

According to another Maxwell equation.

$$\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t} \quad \text{——— ④}$$

Sub eqn ④ in eqn ③.

$$\nabla(\nabla \cdot \bar{E}) - \nabla^2 \bar{E} = -\mu \frac{\partial}{\partial t}\left[\bar{J} + \frac{\partial \bar{D}}{\partial t}\right]$$

For free space $\nabla \cdot \bar{E} = 0$.

$$-\nabla^2 \bar{E} = -\mu\left[\frac{\partial}{\partial t}\left(\bar{J} + \frac{\partial \bar{D}}{\partial t}\right)\right]$$

$$\boxed{\nabla^2 \bar{E} = \mu\left[\frac{\partial}{\partial t}\left(\bar{J} + \frac{\partial \bar{D}}{\partial t}\right)\right]} \quad \text{——— (5)}$$

When any field varies with respect to time, its partial derivative taken with respect to time can be replaced by $j\omega$.

$$\nabla^2 \bar{E} = \mu\left[j\omega\left(\bar{J} + j\omega \bar{D}\right)\right] = j\omega\mu\left[(\sigma \bar{E} + j\omega(\varepsilon_0 \bar{E})\right]$$

$$\nabla^2 \bar{E} = \left[j\omega\sigma\mu \bar{E} + (j\omega)^2 \varepsilon\mu \bar{E}\right]$$

$$\nabla^2 \bar{E} = j\omega\mu(\sigma + j\omega\varepsilon_0)\bar{E} \quad \text{——— (6)}$$

Similarly

$$\nabla^2 \bar{H} = j\omega\mu(\sigma + j\omega\varepsilon_0)\bar{H} \quad \text{——— (7)}$$

The properties of the medium in which wave is propagating. Then

$$\nabla^2 \bar{E} = \gamma^2 \bar{E} \quad \& \quad \nabla^2 \bar{H} = \gamma^2 \bar{H} \quad \text{——— (8)}$$

γ – propagation constant $= \alpha + j\beta = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon_0)}$ ——— (9)

where $\alpha = \omega\sqrt{\frac{\mu\varepsilon_0}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon_0}\right)^2} - 1\right]}$ → real ——— (10)

$$\beta = \omega\sqrt{\frac{\mu\varepsilon_0}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon_0}\right)^2} + 1\right]} \rightarrow \text{imaginary} \quad \text{——— (11)}$$

Intrinsic impedance of the medium.

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon_0}}$$

$$|\eta| = \frac{\sqrt{\mu/\varepsilon_0}}{\sqrt[4]{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2}} \quad \text{and} \quad \tan 2\theta = \frac{\sigma}{\omega\varepsilon_0}$$

$$0^\circ < \theta < 45^\circ$$

Uniform Plane waves in Perfect Dielectric

[lossless]

The medium is perfect dielectrics, then

$\sigma = 0$, $\mu = \mu_r \mu_0$ and $\varepsilon = \varepsilon_0 \varepsilon_r$.

The velocity of propagation is given by

$$v = \frac{1}{\mu \varepsilon} = \frac{1}{\sqrt{\mu_0 \mu_r \times \varepsilon_0 \varepsilon_r}}$$

$$v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}\sqrt{\mu_r \varepsilon_r}} = \frac{1/\sqrt{\mu_0 \varepsilon_0}}{\sqrt{\mu_r \varepsilon_r}}$$

$$v = \frac{c}{\sqrt{\mu_r \varepsilon_r}} \text{ m/s} \quad \text{———} \quad (1)$$

$$v = \frac{1}{\sqrt{\mu \varepsilon}} = \frac{\omega}{\omega\sqrt{\mu \varepsilon}} = \frac{\omega}{\beta} \text{ m/s} \quad \text{———} \quad (2)$$

The propagation constant is given by

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} \quad m^{-1}$$

$$\gamma = +j\omega\sqrt{\mu\varepsilon} \quad m^{-1}$$

$$\gamma = \alpha + j\beta.$$

where α – attenuation constant $= 0$

$\beta \rightarrow$ phase constant $= \omega\sqrt{\mu\varepsilon}$ rad/sec.

Intrensic impedence:

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$$

$\sigma = 0$.

$$\eta = \sqrt{\frac{\mu_0}{\varepsilon_0}}\sqrt{\frac{\mu_r}{\varepsilon_r}} = \eta_0\sqrt{\frac{\mu_r}{\varepsilon_r}}$$

$$\boxed{\eta = \eta_0\sqrt{\frac{\mu_r}{\varepsilon_r}}}$$

Uniform Plane Waves in Lossy Dielectric

Due to certain conductivity, certain amount of loss in the medium takes place. Hence the wave travelling through such medium gets attenuated ($\alpha \neq 0$) such dielectric is called lossy dielectric.

Let us consider that a uniform plane wave travels in x-direction through the lossy dielectric $\alpha \neq 0$. The wave equation for the electric field vector $\bar{E}$ is given by.

$$\nabla^2 \bar{E} = \mu \sigma \frac{\partial \bar{E}}{\partial t} + \mu \varepsilon \frac{\partial^2 \bar{E}}{\partial t^2} \quad \text{——— ①}$$

Using vector identity.

$$\frac{\partial^2 \bar{E}}{\partial x^2} + \frac{\partial^2 \bar{E}}{\partial y^2} + \frac{\partial^2 \bar{E}}{\partial z^2} = \mu \sigma \frac{\partial \bar{E}}{\partial t} + \mu \varepsilon \frac{\partial^2 \bar{E}}{\partial t^2} \quad \text{——— ②}$$

The wave travels in x-direction,

$$\frac{\partial^2 \bar{E}}{\partial x^2} = \mu \sigma \frac{\partial \bar{E}}{\partial t} + \mu \varepsilon \frac{\partial^2 \bar{E}}{\partial t^2} \quad \text{——— ③}$$

Let E has only one component ie in X-direction. (E_x).

$$\frac{\partial^2 \bar{E}_x}{\partial x^2} = \mu \sigma \frac{\partial \bar{E}_x}{\partial t} + \mu \varepsilon \frac{\partial^2 \bar{E}_x}{\partial t^2}$$

Ex in phasor form,

$$E_x = E_m e^{j\omega t}$$

$$\frac{\partial E_x}{\partial t} = j\omega E_m e^{j\omega t} = (j\omega) E_x \quad \text{——— ④}$$

$$\frac{\partial^2 E_x}{\partial t^2} = (j\omega)(j\omega) E_m e^{j\omega t} = -\omega^2 E_x \quad \text{——— ⑤}$$

Sub. value ④ & ⑤ in above eqn.

$$\frac{\partial^2 E_x}{\partial z^2} = \mu \sigma (j\omega) E_x + \mu \varepsilon_0 (j\omega)^2 E_x \quad \text{—— ⑥}$$

$$\frac{\partial^2 E_x}{\partial z^2} = \left[j\omega\mu (\sigma + j\omega \varepsilon) \right] E_x. \quad \text{—— ⑦}$$

W.K.T Propagation constant $\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$

Sub in eqn ⑦.

$$\frac{\partial^2 E_x}{\partial z^2} = \gamma^2 E_x$$

$$\frac{\partial^2 E_x}{\partial z^2} - \gamma^2 E_x = 0 \quad \text{—— ⑧}$$

Let $\frac{\partial}{\partial z} = D$, $\frac{\partial^2}{\partial^2 z} = D^2$, then

$$D^2 E_x - \gamma^2 E_x = 0$$

$$E_x (D^2 - \gamma^2) = 0$$

$$D^2 - \gamma^2 = 0$$

$$D^2 = \gamma^2$$

$$D = \pm \gamma$$

and

$$E_x = k_1 e^{-\gamma z} + k_2 e^{+\gamma z}$$

$K_1, K_2 \rightarrow$ constant with respect to z

$$K_1 = E_m^+ e^{j\omega t}$$

$$K_2 = E_m^- e^{j\omega t}$$

then

$$E_x = E_m^+ e^{j\omega t} \cdot e^{-\gamma z} + E_m^- e^{j\omega t} e^{\gamma z}$$

$$\gamma = \alpha + j\beta$$

$$\therefore \boxed{E_x = E_m^+ e^{-\alpha z} \cdot e^{j(\omega t - \beta z)} + E_m^- e^{\alpha z} \cdot e^{j(\omega t + \beta z)}}$$

Assume, the wave travels in x-direction, then

$$E_x = E_m^+ \, e^{-\alpha z} \, e^{j(\omega t - \beta z)}$$

$$= E_m^+ \, e^{-\alpha x} \cos(\omega t - \beta x)$$

The wave travels in positive x-direction with velocity ω/β m/s. As the wave progresses, it gets attenuated by the factor $e^{-\alpha}$. Thus α - indicates the rate at which wave is attenuated in amplitude during the propagation through lossy dielectric with some finite non-zero conductivity.

As $\sigma \neq 0$, the intrinsic impedence becomes a complex quantity.

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = |\eta| \angle \theta_n \ \Omega$$

$$\theta_n = \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{\omega\varepsilon}{\sigma}\right)\right] \text{ rad}.$$

For very low frequency signal $\theta_n \approx \pi/4$

high frequency signal $\theta_n = 0$

<u>Uniform Plane Wave in good conductor</u>

The good conductor has very high conductivity (10^7 ℧).

Ex. copper, aluminium etc.

For good conductors,

$$\frac{\sigma}{\omega\varepsilon} \gg 1$$

Propagation constant

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$$

As $\sigma >> \omega\epsilon$, neglect imarg. part

$$\gamma = \sqrt{j\omega\mu\sigma} = \sqrt{j}\sqrt{\omega\mu\sigma}$$

$$j = 1\angle 90^\circ$$

$$\gamma = \sqrt{\omega\mu\sigma}\sqrt{1\angle 90^\circ} = \sqrt{\omega\mu\sigma}\ \angle 45^\circ$$

$$= \sqrt{\omega\mu\sigma}\ \cos 45^\circ + j\sin 45^\circ$$

$$= \sqrt{\omega\mu\sigma}\left[\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right]$$

$$= \sqrt{2\pi f\mu\sigma}\left[\frac{1}{\sqrt{2}}(1+j1)\right]$$

$$\gamma = \alpha + j\beta = \sqrt{\pi f\mu\sigma} + j\sqrt{\pi f\mu\sigma}$$

Good conductor,

$$\alpha = \sqrt{\pi f\mu\sigma} \quad NP/m$$

$$\beta = \sqrt{\pi f\mu\sigma} \quad rad/m$$

The intrensic impedence of a good conductor

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}$$

$$\sigma >> j\omega\epsilon$$

$$\therefore \quad \eta = \sqrt{\frac{j\omega\mu}{\sigma}} = \sqrt{\frac{\omega\mu}{\sigma}}\sqrt{j}$$

$$\sqrt{j} = \sqrt{1\angle 90} = 1\angle 45^\circ = \cos 45^\circ + j\sin 45^\circ$$

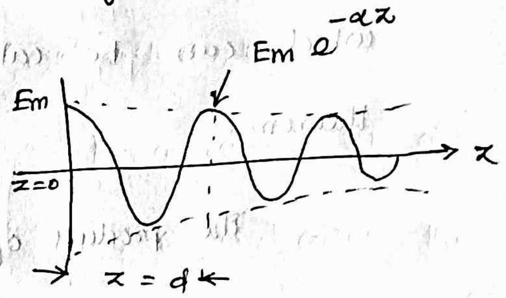
$$\sqrt{j} = \left(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right)$$

$$\therefore \quad \eta = \sqrt{\frac{\omega\mu}{\sigma}}\left[\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right]$$

$$\therefore \quad \boxed{\eta = \sqrt{\frac{\omega\mu}{2}}\ (1+j1)}$$

consider only the component of Electric field Ex travelling in positive x- direction, when it travels in good conductor, the conductivity is very high and α also very high.

$$E_x = E_m^+ e^{-\alpha x} e^{-j\beta x}$$



The distance through which the amplitude of the travelling wave decreases to 37% of the oregional amplitude is called skin depth (or) depth of penetration

$$\text{skin depth } \delta = \frac{1}{\alpha} = \frac{1}{\beta} = \frac{1}{\sqrt{\pi f \mu \sigma}} \text{ m}$$

The intrensic impedence,

$$\eta = \frac{1}{\sigma\delta} + j\frac{1}{\sigma\delta} = \frac{\sqrt{2}}{\sigma\delta} \angle 45^\circ$$

$$\text{velocity } v = \frac{\omega}{\beta} = \frac{\sqrt{2}(\sqrt{\omega})^2}{\sqrt{\omega\mu\sigma}}$$

$$v = \sqrt{\frac{2\omega}{\mu\sigma}} = \omega\delta \text{ m/s.}$$

$$\text{wave length } \lambda = \frac{2\pi}{\beta} = 2\pi\delta \text{ m.}$$

Poynting Vector & Poynting Theorem

The energy stored in electric field and magnetic field is transmitted at a certain rate of energy flow which can be calculated with the help of poynting theorem.

The product of $\bar{E}$ and $\bar{H}$ gives a new quantity which is expressed as watt per unit area.

$\bar{E}$ unit V/m

$\bar{H}$ unit A/m

$$\times \quad \frac{V}{m} \times \frac{A}{m} = \frac{VA}{m^2} = \frac{\text{Watts}}{\text{Area}}$$

This quantity is called power density.

Statement:-

The vector product of electric field intensity $\bar{E}$ and magnetic field intensity ($\bar{H}$) at any point is a measure of the rate of energy flow per unit area at the point and the direction of power flow is perpendicular to $\bar{E}$ and $\bar{H}$ both along the direction of $\bar{E} \times \bar{H}$.

$$\bar{P} = \bar{E} \times \bar{H}$$

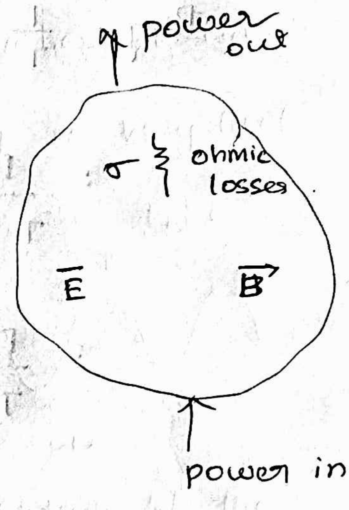
$\bar{P} \rightarrow$ poynting vector.

law of conservation,

The net power flowing out of a given volume V is equal to the time rate of decrease in the energy stored within volume

V minus the ohmic power dissipated.

$$\bar{E} = E_x \, \bar{a_x}$$
$$\bar{H} = H_y \, \bar{a_y}$$



$$\bar{P} = \bar{E} \times \bar{H}$$
$$= (E_x \, \bar{a_x}) \times H_y \, \bar{a_y}$$
$$= E_x H_y \, \bar{a_z}$$
$$= P_z \, \bar{a_z}$$

consider that the electric field propagates in free space is given by

$$\bar{E} = [E_m \cos(\omega t - \beta z)] \, \bar{a_x}$$

$$\eta = \eta_0 = \frac{E_m}{H_m} = 120\pi = 377 \, \Omega$$

In free space, electromagnetic waves travels at a speed of light.

$$\bar{H} = [E_m \cos(\omega t - \beta z)] \, \bar{a_y}$$

$$\bar{H} = \left[\frac{E_m}{\eta_0} \cos(\omega t - \beta z)\right] \bar{a_y}$$

According to poynting theorem,

$$\bar{P} = \bar{E} \times \bar{H}$$

$$\bar{P} = \left[E_m \cos(\omega t - \beta z) \, \bar{a_x}\right] \times \left[\frac{E_m}{\eta_0} \cos(\omega t - \beta z) \, \bar{a_y}\right]$$

$$\boxed{\bar{P} = \frac{E_m^2}{\eta_0} \cos^2(\omega t - \beta z) \, \bar{a_z} \quad W/m^2}$$

//.

Average power density:-

$$P_{avg} = \frac{1}{T}\int_0^T \frac{E_m^2}{\eta}\cos(\omega t - \beta z)\,dt$$

$$= \frac{E_m^2}{T\eta}\int_0^T \frac{1+\cos 2(\omega t - \beta z)}{2}\,dt$$

$$= \frac{E_m^2}{T\eta}\left[\frac{t}{2} + \frac{\sin 2(\omega t - \beta z)}{2\times 2\omega}\right]_0^T$$

$$= \frac{E_m^2}{T\eta}\left[\frac{T}{2} + \frac{\sin 2\omega T - 2\beta z}{4\omega} - \frac{\sin(-2\beta z)}{4\omega}\right]$$

But $\omega T = 2\pi$

$$P_{avg} = \frac{E_m^2}{T\eta}\left[\frac{T}{2} + \frac{\sin(4\pi - 2\beta z)}{4\omega} + \frac{\sin 2\beta z}{4\omega}\right]$$

$$= \frac{E_m^2}{\cancel{T}\eta}\left[\frac{\cancel{T}}{2} - \cancel{\frac{\sin 2\beta z}{4\omega}} + \cancel{\frac{\sin 2\beta z}{4\omega}}\right]$$

$$\boxed{P_{avg} = \frac{E_m^2}{2\eta}} \; W/m^2$$